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Continuity of the dynamics in a localized large diffusion problem with nonlinear boundary conditions

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ABSTRACT

This paper is concerned with singular perturbations in parabolic problems subjected to nonlinear Neumann boundary conditions. We consider the case for which the diffusion coefficient blows up in a subregion Ω_0 which is interior to the physical domain $\Omega \subset \mathbb{R}^n$. We prove, under natural assumptions, that the associated attractors behave *continuously* as the diffusion coefficient blows up locally uniformly in Ω_0 and *converges* uniformly to a continuous and positive function in $\Omega_1 = \bar{\Omega} \setminus \bar{\Omega}_0$.

1. Introduction

In this paper we are concerned with the asymptotic dynamics of a reaction–diffusion equation with nonlinear Neumann boundary conditions with varying diffusivity. We consider the situation for which the diffusivity becomes very large in a localized interior region of the physical domain whereas it remains bounded and bounded away from zero in the remaining part of the domain. This situation can be found, for example, in composite materials, where the heat diffusion properties may differ significantly from one part of the material to another.

In order to introduce precisely the results that we wish to prove in this paper let us introduce some terminology (following closely [17]), let $\Omega \subset \mathbb{R}^n$ be a bounded domain with smooth boundary $\Gamma = \partial\Omega$, $\Omega_0 \subset \Omega$ be an open domain with smooth boundary Γ_0 such that $\bar{\Omega}_0 \subset \Omega$ and $\Omega_1 = \Omega \setminus \bar{\Omega}_0$. Assume that there is a positive integer m such that $\Omega_0 = \bigcup_{i=1}^m \Omega_{0,i}$, where $\Omega_{0,i}$ is a smooth, connected sub-domain of Ω_0, $\bar{\Omega}_{0,i} \cap \bar{\Omega}_{0,j} = \varnothing$, for $i \neq j$. Notice that $\partial\Omega_1 = \Gamma \cup \Gamma_0$ and, if $\Gamma_{0,i} = \partial\Omega_{0,i}$, then $\Gamma_0 = \bigcup_{i=1}^m \Gamma_{0,i}$. Let $\epsilon_0 > 0$ and $\epsilon \in (0, \epsilon_0]$ be a parameter.

Assume that, for some $m_0 > 0$, $p_\epsilon \in \mathcal{C}^1(\bar{\Omega}, [m_0, \infty))$ for every $x \in \Omega$ and $0 < \epsilon \leqslant \epsilon_0$. Also, assume that p_ϵ satisfies

$$p_\epsilon(x) \to \begin{cases} p(x) & \text{uniformly on } \Omega_1, \\ \infty & \text{uniformly on compact subsets of } \Omega_0, \end{cases}$$

where $p \in \mathcal{C}^1(\bar{\Omega}_1, [m_0, \infty))$.

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For $\lambda > 0$ we consider the family of parabolic equations

$$\begin{cases} u_t^\epsilon - \operatorname{div}\bigl(p_\epsilon(x)\nabla u^\epsilon\bigr) + \lambda u^\epsilon = f\bigl(u^\epsilon\bigr) & \text{in } \Omega, \\ \dfrac{\partial u^\epsilon}{\partial \vec{n}} = g\bigl(u^\epsilon\bigr) & \text{on } \partial\Omega, \\ u^\epsilon(0) = u_0^\epsilon, \end{cases} \tag{1.1}$$

where the nonlinearities f and g satisfy the growth (**G**) and sign (**S**) conditions as follows:

(**G**) If $N = 2$, for every $\eta > 0$ there is a $c_\eta > 0$ such that, for $j = f, g$,

$$\bigl|j(u) - j(v)\bigr| \leqslant c_\eta\bigl(e^{\eta|u|^2} + e^{\eta|v|^2}\bigr)|u - v|, \quad \forall u, v \in \mathbb{R},$$

and if $N \geqslant 3$, there is a constant $c > 0$ such that

$$\bigl|f(u) - f(v)\bigr| \leqslant c|u - v|\bigl(|u|^{\frac{4}{N-2}} + |v|^{\frac{4}{N-2}} + 1\bigr),$$

$$\bigl|g(u) - g(v)\bigr| \leqslant c|u - v|\bigl(|u|^{\frac{2}{N-2}} + |v|^{\frac{2}{N-2}} + 1\bigr),$$

for any $u, v \in \mathbb{R}$.

(**S**) Assume that there exist $B_0, C_0 \in \mathbb{R}$ and $B_1, C_1 \geqslant 0$ such that

$$uf(u) \leqslant -C_0u^2 + C_1|u|,$$

$$ug(u) \leqslant -B_0u^2 + B_1|u|,$$

for all $u \in \mathbb{R}$.

For such problems a spatial homogenization process occurs in the region Ω_0 in which the diffusion is large. Proceeding as in [17], if u^ϵ *converges* to u as $\epsilon \to 0$ and u takes a time dependent spatially constant value $u_{\Omega_0}(t)$ in Ω_0 we heuristically obtain that the limiting problem should be

$$\begin{cases} u_t - \operatorname{div}\bigl(p(x)\nabla u\bigr) + \lambda u = f(u) & \text{in } \Omega_1, \\ u_{|\Omega_{0,i}} := u_{\Omega_{0,i}} & \text{in } \Omega_{0,i},\ i = 1, \dots, m, \\ \dot{u}_{\Omega_{0,i}} + \dfrac{1}{|\Omega_{0,i}|} \displaystyle\int_{\Gamma_{0,i}} p(x)\frac{\partial u}{\partial \vec{n}}\,dx + \lambda u_{\Omega_{0,i}} = f(u_{\Omega_{0,i}}) & i = 1, \dots, m, \\ \dfrac{\partial u}{\partial \vec{n}} = g(u) & \text{on } \Gamma, \\ u(0) = u_0. \end{cases} \tag{1.2}$$

With an appropriate functional analytic framework, we can write the problems (1.1) and (1.2) as an abstract evolution equation

$$\begin{cases} \dot{u} + A_\epsilon u = h_\epsilon(u), \\ u(0) = u_0 \in X_\epsilon^\alpha. \end{cases} \tag{1.3}$$

The well posedness of (1.3) has been established in [5,7,10].

In this paper we are interested on the study of the continuity of the family of global attractors $\{\mathcal{A}_\epsilon\colon \epsilon \in [0, \epsilon_0]\}$ associated to (1.3) at $\epsilon = 0$. The upper semicontinuity of attractors for problems with localized large diffusion has been considered in [7] in $H^1(\Omega)$ and $C^0(\bar{\Omega})$ topologies. Also, for the case $n = 1$ the continuity has been proved in [9], where the authors prove that the limiting problem is generically Morse–Smale and, using the fact that the spectra of the associate linear operators posses large gaps, C^1-convergence of the invariant manifolds leading topological equivalence of the family of attractors for small values of ϵ. For the case of homogeneous Dirichlet boundary conditions, the upper and lower semicontinuity of attractors as ϵ tends to zero is studied in [8]. Here we extend these results to the case of nonlinear Neumann boundary conditions.

In a general sense, the upper semicontinuity of the attractors can be obtained for a large class of dissipative problems, as in [12]. Lower semicontinuity, however, is a more delicate matter and usually is obtained for gradient semigroups. The pioneer works to establish the procedure to obtain the lower semicontinuity of attractors for gradient semigroups were [13] and [14].

To obtain that the semigroup generated by Eq. (1.3) with $\epsilon = 0$ has a global attractor, we have to assume the dissipativeness assumption (see [6] for a related condition):

($\mathbf{D}_0$) *Suppose that the sign condition* ($\mathbf{S}$) *holds and that, for C_0 and B_0 from* ($\mathbf{S}$)*, the first eigenvalue μ_1 of the limiting problem*

$$\begin{cases} -\operatorname{div}\big(p_0(x)\nabla u\big)+(\lambda+C_0)u=\mu u & \text{in } \Omega_1,\\ u_{|\Omega_{0,i}}:=u_{\Omega_{0,i}} & \text{in } \Omega_{0,i},\ i=1,\dots,m,\\ \dfrac{1}{|\Omega_{0,i}|}\displaystyle\int_{\Gamma_{0,i}} p_0(x)\frac{\partial u}{\partial\vec n}\,\mathrm{d}x+(\lambda+C_0)u_{\Omega_{0,i}}=\mu u_{\Omega_{0,i}} & i=1,\dots,m,\\ p_0(x)\dfrac{\partial u}{\partial\vec n}+B_0u=0 & \text{on } \Gamma \end{cases} \tag{1.4}$$

is positive.

The existence of a global attractor for (1.3) with $\epsilon\in(0,\epsilon_0)$ will result from spectral convergence. In fact, if the first eigenvalue of the limiting problem (1.4) is positive, spectral convergence (see Section 3) will imply that, for $\epsilon>0$ suitably small, the first eigenvalue of problem

$$\begin{cases} -\operatorname{div}\big(p_\epsilon(x)\nabla u^\epsilon\big)+(\lambda+C_0)u^\epsilon=\mu u^\epsilon & \text{in } \Omega,\\ p_\epsilon(x)\dfrac{\partial u^\epsilon}{\partial\vec n}+B_0u^\epsilon=0 & \text{on } \Gamma \end{cases} \tag{1.5}$$

is also positive. This fact will give us a dissipativeness assumption for the perturbed problem which in turn leads to existence of global attractors (see [6]).

To prove the continuity of the family of attractors we follow a program which has been used in many works and different perturbation problems. For instance, concerning to singular perturbations of the domains, we can cite [4] for Dumbbell type domains and [18], where this agenda is applied for examples on thin domains. We also refer [2] for a highly oscillating boundary problem, [8] for localized large diffusion problems with homogeneous Dirichlet boundary conditions and [11] for a general scheme developed to treat the continuity of the attractors of semilinear parabolic problems.

This program is based in a careful study of the behavior of the linear parts under the perturbation. It consists of: (1) Determining the appropriate functional analytic framework for the associated perturbation problem and introducing the appropriate notion of convergence; (2) Studying the convergence of resolvent operators; (3) Obtaining a type of Trotter–Kato Theorem to ensure the convergence of the associated linear semigroups; (4) Proving the convergence of nonlinear semigroups using the Variation of Constants Formula; (5) Establishing the upper semicontinuity of attractors; (6) Proving the continuity of the set of equilibria (assuming that all equilibria of the limiting problem are hyperbolic); (7) Proving the continuity of the linearized semigroups and of the associated linear unstable manifolds; (8) Proving the continuity of the nonlinear local unstable manifolds as graphs and (9) Proving the lower semicontinuity of the global unstable sets and of global attractors.

With this in mind, the paper is organized as follows. In Section 2, we introduce the appropriate functional setting to treat the problems (1.3) and we recall (from [6,10]) known results on the well posedness, regularity and existence of bounded global attractors. In Section 3 we establish the basic properties of the linear operators A_ϵ and A_0; in particular, we present the fundamental result on compact convergence of the resolvent operators that will lead to the continuity of eigenvalues and eigenfunctions of the associated linear operators. We also establish a version of Trotter–Kato Theorem for linear semigroups. In Section 4, we obtain the upper semicontinuity of attractors working with the usual script: it follows from the continuity of nonlinear semigroups which in turn is obtained using the convergence of linear semigroups and Variation of Constants Formula. To show the lower semicontinuity of attractors, we assume that all equilibrium solutions of (1.3) with $\epsilon=0$ are hyperbolic and prove the continuity of the set of equilibria. The continuity of the set of equilibria implies the continuity of the linear unstable manifold of the linearization around equilibria. Using the continuity of the linear unstable manifolds, we obtain the continuity of the nonlinear local (near the equilibrium) unstable manifolds, from which it the lower semicontinuity of global unstable sets and of global attractors follows.

2. Functional setting and background results

In this section we write (1.1) and (1.2) in an abstract semilinear form and introduce an appropriate functional analytic framework to study them. First we present some results on existence, uniqueness and regularity of solutions of reaction–diffusion equations with nonlinear boundary conditions, following [10]. We also establish existence and uniform bounds of attractors for the semigroups associated to these problems (see [6,10] for details).

Consider the usual Sobolev spaces $\mathrm{H}^s(\Omega)$, $\mathrm{W}^{s,p}(\Omega)$, $s\geqslant 0$ and the trace spaces $\mathrm{H}^\sigma(\Gamma)$, $\mathrm{W}^{\sigma,p}(\Gamma)$, $\sigma\geqslant 0$. Denote by $\mathrm{H}^{-s}(\Omega)$ the dual space of $\mathrm{H}^s(\Omega)$. The duality pairing between these spaces will be denoted by $\langle\cdot,\cdot\rangle_{-s,s}$. Also, $\langle\cdot,\cdot\rangle_\Omega$ will denote the inner product in $\mathrm{L}^2(\Omega)$ and $\langle\cdot,\cdot\rangle_\Gamma$ the inner product in $\mathrm{L}^2(\Gamma)$.

The trace operator defined in $\mathrm{H}^s(\Omega)$, with values in $\mathrm{H}^{s-\frac{1}{2}}(\Gamma)$, for $s > \frac{1}{2}$, will be denoted by γ. Furthermore, for a function $u \in \mathrm{H}^s(\Omega)$, we identify its trace $\gamma(u) \in \mathrm{H}^{s-\frac{1}{2}}(\Gamma) \subset \mathrm{H}^{-s}(\Omega)$ with the linear form $\gamma(u) \in \mathrm{H}^{-s}(\Omega)$ in the following manner, for any $\phi \in \mathrm{H}^s(\Omega)$,

$$\big\langle \gamma(u), \phi \big\rangle_{-s,s} := \langle u, \phi \rangle_\Gamma := \int_\Gamma \gamma(u)\gamma(\phi)\,\mathrm{d}x.$$

Recall that, from Trace Theorem, $\gamma : \mathrm{W}^{s,q}(\Omega) \to \mathrm{W}^{s-\frac{1}{q},q}(\Gamma)$, $s > \frac{1}{q}$, is a bounded linear transformation.

Also consider the normal derivative operator, relative to the operator $-\operatorname{div}(p_\epsilon(x)\nabla u)$; that is, if

$$u \in Z := \big\{ z \in \mathrm{H}^1(\Omega)\colon\ -\operatorname{div}\big(p_\epsilon(x)\nabla z\big) \in \mathrm{L}^2(\Omega) \big\},$$

then $\frac{\partial u}{\partial \vec{n}} \in \mathrm{H}^{-\frac{1}{2}}(\Gamma)$ and it is defined as

$$\begin{aligned}
\left\langle \frac{\partial u}{\partial \vec{n}}, \gamma(v) \right\rangle_{-\frac{1}{2},\frac{1}{2}} &= -\int_\Omega -\operatorname{div}\big(p_\epsilon(x)\nabla u\big) v\,\mathrm{d}x + \int_\Omega p_\epsilon(x)\nabla u \nabla v\,\mathrm{d}x \\
&= \int_\Omega \operatorname{div}\big(p_\epsilon(x) v \nabla u\big)\,\mathrm{d}x, \qquad (2.6)
\end{aligned}$$

for every $v \in \mathrm{H}^1(\Omega)$.

Under these conditions and assuming that $\lambda > 0$, we introduce the canonical isometric isomorphism between $\mathrm{H}^1(\Omega)$ and its dual, $\mathrm{H}^{-1}(\Omega)$, such that, for every $u, \phi \in \mathrm{H}^1(\Omega)$,

$$\langle A_\epsilon u, \phi \rangle_{-1,1} = \int_\Omega p_\epsilon(x)\nabla u \nabla \phi\,\mathrm{d}x + \lambda \int_\Omega u\phi\,\mathrm{d}x.$$

In this way, we can rewrite (2.6) (for $u \in Z$) as

$$\langle A_\epsilon u, v \rangle_{-1,1} = \big\langle -\operatorname{div}\big(p_\epsilon(x)\nabla u\big) + \lambda u, v \big\rangle_\Omega + \left\langle \frac{\partial u}{\partial \vec{n}}, \gamma(v) \right\rangle_{-\frac{1}{2},\frac{1}{2}}. \qquad (2.7)$$

We also consider in $\mathrm{H}^1(\Omega)$ the inner product

$$a_\epsilon(u, v) = \int_\Omega p_\epsilon(x)\nabla u \nabla v\,\mathrm{d}x + \lambda \int_\Omega uv\,\mathrm{d}x = \langle A_\epsilon u, v \rangle_{-1,1},$$

which gives a norm in $\mathrm{H}^1(\Omega)$, equivalent to the usual one.

Now, if u is a solution of (1.1), multiplying (1.1) by $\varphi \in \mathrm{H}^1(\Omega)$ and integrating by parts we have that

$$\int_\Omega u_t\varphi\,\mathrm{d}x + \int_\Omega p_\epsilon(x)\nabla u \nabla \varphi\,\mathrm{d}x + \lambda \int_\Omega u\varphi\,\mathrm{d}x = \int_\Omega f(u)\varphi\,\mathrm{d}x + \int_\Gamma g\big(\gamma(u)\big)\gamma(\varphi)\,\mathrm{d}x.$$

Notice that

$$\big\langle g\big(\gamma(u), \gamma(\varphi)\big)\big\rangle_\Gamma = \int_\Gamma g\big(\gamma(u)\big)\gamma(\varphi)\,\mathrm{d}x = \int_\Gamma \gamma\big(g(u)\gamma(\varphi)\big)\,\mathrm{d}x = \big\langle \gamma\big(g(u)\big), \gamma(\varphi)\big\rangle_{-1,1}.$$

Then,

$$\langle u_t, \varphi \rangle_{-1,1} + \langle A_\epsilon u, \varphi \rangle_{-1,1} = \big\langle f(u), \varphi \big\rangle_{-1,1} + \big\langle g\big(\gamma(u)\big), \gamma(\varphi) \big\rangle_{-1,1}.$$

The part of A_ϵ in $\mathrm{L}^2(\Omega)$ (denoted the same) is the operator $A_\epsilon : \mathcal{D}(A_\epsilon) \subset \mathrm{L}^2(\Omega) \to \mathrm{L}^2(\Omega)$ defined by

$$\begin{aligned}
&\mathcal{D}(A_\epsilon) = \left\{ u \in \mathrm{H}^1(\Omega)\colon\ -\operatorname{div}\big(p_\epsilon(x)\nabla u\big) \in \mathrm{L}^2(\Omega) \text{ and } \frac{\partial u}{\partial \vec{n}} = 0 \text{ on } \Gamma \right\}, \\
&A_\epsilon u = -\operatorname{div}\big(p_\epsilon(x)\nabla u\big) + \lambda u, \quad u \in \mathcal{D}(A_\epsilon).
\end{aligned}$$

As in [17], $X_0 = \mathrm{L}^2_{\Omega_0}(\Omega) := \{u \in X\colon\ u \text{ is constant in } \Omega_0\}$, $\mathrm{H}^1_{\Omega_0}(\Omega) := \{u \in \mathrm{H}^1(\Omega)\colon\ \nabla u = 0 \text{ in } \Omega_0\}$ and $a_0 : \mathrm{H}^1_{\Omega_0}(\Omega) \times \mathrm{H}^1_{\Omega_0}(\Omega) \to \mathbb{R}$ the bilinear form given by

$$a_0(u, v) = \int_{\Omega_1} p_0(x)\nabla u \nabla v\,\mathrm{d}x + \lambda \int_\Omega uv\,\mathrm{d}x,$$

and $A_0 : \mathcal{D}(A_0) \subset L^2_{\Omega_0}(\Omega) \to L^2_{\Omega_0}(\Omega)$ the operator defined by

$$\mathcal{D}(A_0) = \left\{ u \in H^1_{\Omega_0}(\Omega)\colon\ -\operatorname{div}\bigl(p_0(x)\nabla u\bigr) \in L^2(\Omega_1) \text{ and } \frac{\partial u}{\partial \vec{n}} = 0 \text{ on } \Gamma \right\},$$

$$A_0 u = \bigl(-\operatorname{div}\bigl(p_0(x)\nabla u\bigr) + \lambda u\bigr)\chi_{\Omega_1} + \sum_{i=1}^{m} \frac{1}{|\Omega_{0,i}|}\left(\int_{\Gamma_{0,i}} p_0(x)\frac{\partial u}{\partial \vec{n}}\,dx + \int_{\Omega_{0,i}} \lambda u_{\Omega_{0,i}}\,dx\right)\chi_{\Omega_{0,i}}, \quad u \in \mathcal{D}(A_0),$$

where χ_B is the characteristic function of the set B.

Note that $A_\epsilon - \lambda I$, $\epsilon \in [0,\epsilon_0]$, is a self-adjoint and nonnegative operator. Consequently A_ϵ is a sectorial operator which generates exponentially decaying analytic semigroup $\{e^{-A_\epsilon t}\colon\ t \geqslant 0\}$. Denote by A_ϵ^α the fractional power operators associated to A_ϵ (see [1,15]). Let $X_\epsilon^\alpha = D(A_\epsilon^\alpha)$ with the graph norm and $X_\epsilon^{-\alpha} = (X_\epsilon^\alpha)'$, $\alpha \geqslant 0$, $\epsilon \in [0,\epsilon_0]$. Furthermore, for $1 \geqslant \beta \geqslant \alpha \geqslant -1$,

$$\bigl\|e^{-A_\epsilon t}\bigr\|_{\mathcal{L}(X_\epsilon^\alpha, X_\epsilon^\beta)} \leqslant M_\omega t^{\alpha-\beta} e^{-\omega t}, \quad \omega < \lambda, \text{ for all } t > 0, \tag{2.8}$$

with M_ω independent of ϵ.

Remark 2.1. From the fact that the operators A_ϵ are self-adjoint with numerical range contained in $(-\infty, -\lambda]$ it follows from Theorem 1.3.9 in [16] that, uniform with respect to ϵ, estimates on the resolvent operators can be obtained on any closed sector that does not intersect $(-\infty, -\lambda]$. From Theorems 1.3.4 and 1.4.3 in [15] the uniformity of M_ω with respect to ϵ follows. Recall that the negative powers correspond to positive powers for the realization of A_ϵ in X_ϵ^{-1}, see [1].

To obtain appropriate semilinear formulations for (1.1) and (1.2) we consider nonlinear maps, in the form $h(u) := f_\Omega(u) + g_\Gamma(u) \in H^{-1}(\Omega)$ with $\langle h(u), \phi\rangle = \langle f_\Omega(u), \phi\rangle_\Omega + \langle g_\Gamma(u), \phi\rangle_\Gamma$.

We wish to consider

$$h : X_\epsilon^\alpha \to X_\epsilon^\beta$$

for $\beta < 0 < \alpha$ and $\alpha - \beta \leqslant 1$ suitably chosen. To obtain a semigroup in $H^1(\Omega)$ we fix $\alpha = \frac{1}{2}$. Since $g \neq 0$, we need $\beta < 0$ and we must have that $-\frac{1}{2} \leqslant \beta < 0$. Moreover, notice that in the case of non-zero terms on the boundary, there is another natural upper bound for β. In fact, $\beta + 1 < \frac{3}{4}$, since for $\beta + 1 \geqslant \frac{3}{4}$, the space $X^{\beta+1}$ incorporates the boundary condition $\frac{\partial u}{\partial \vec{n}} = 0$; that is, $\beta < -\frac{1}{4}$. Summarizing, we must have that $\alpha = \frac{1}{2}$ and $-\frac{1}{2} \leqslant \beta < -\frac{1}{4}$.

With these considerations we write (1.1) and (1.2) abstractly as

$$\begin{cases} \dot{u}^\epsilon + A_\epsilon u^\epsilon = h\bigl(u^\epsilon\bigr), \\ u^\epsilon(0) = u_0^\epsilon \in X_\epsilon^{\frac{1}{2}}, \quad \epsilon \in [0,\epsilon_0], \end{cases} \tag{2.9}$$

where the nonlinearity $h = f_\Omega + g_\Gamma : X_\epsilon^{\frac{1}{2}} \to X_\epsilon^{-\frac{s}{2}}$, $\epsilon \in [0,\epsilon_0]$, is defined by

$$\bigl\langle f_\Omega(u) + g_\Gamma(u), \phi\bigr\rangle = \int_\Omega f(u)\phi\,dx + \int_\Gamma g\bigl(\gamma(u)\bigr)\gamma(\phi)\,dx, \quad \forall \phi \in H^s(\Omega),\ \frac{1}{2} < s \leqslant 1.$$

With these considerations and supposing that f, g satisfy the growth and sign conditions (**G**) and (**S**), we can follow the results from [5] to guarantee that the problems (2.9) are globally well posed in $X_\epsilon^{\frac{1}{2}}$, $\epsilon \in [0,\epsilon_0]$.

That is, for any $u_0^\epsilon \in X_\epsilon^{\frac{1}{2}}$ and $\epsilon \in [0,\epsilon_0]$, the solution $u^\epsilon(t,u_0^\epsilon)$ of (2.9) starting at u_0^ϵ exists for all $t \geqslant 0$. Therefore, we can define in $X_\epsilon^{\frac{1}{2}}$ the nonlinear semigroup $\{T_\epsilon(t)\colon\ t \geqslant 0\}$ associated to (2.9), $\epsilon \in [0,\epsilon_0]$. To simplify we will denote the solution $u^0(t,u_0^0)$ by $u(t,u_0)$.

We are now able to ensure the existence of a global attractor for the limiting problem (2.9) with $\epsilon = 0$. The following result holds (see [7]).

Theorem 2.1. *If* (**G**), (**S**) *and* ($\mathbf{D}_0$) *hold, then the semigroup* $\{T(t)\colon\ t \geqslant 0\}$ *associated to* (2.9) *with* $\epsilon = 0$ *has a global attractor* $\mathcal{A}_0$ *in* $X_0^{\frac{1}{2}}$.

Moreover, this attractor is bounded in $L^\infty(\Omega)$ and the following result holds (see [7]).

Proposition 2.1. *Assume that* (**G**) *and* ($\mathbf{D}_0$) *hold. Denote by* ϕ *the solution of*

$$\begin{cases} -\operatorname{div}\big(p_0(x)\nabla u\big)+(\lambda+C_0)u=C_1 & \text{in } \Omega_1,\\ u_{|\Omega_{0,i}}:=u_{\Omega_{0,i}} & \text{in } \Omega_{0,i},\ i=1,\dots,m,\\ \dfrac{1}{|\Omega_{0,i}|}\displaystyle\int_{\Gamma_{0,i}} p_0(x)\frac{\partial u}{\partial \vec n}\,\mathrm{d}x+(\lambda+C_0)u_{\Omega_{0,i}}=C_1 & i=1,\dots,m,\\ p_0(x)\dfrac{\partial u}{\partial \vec n}+B_0u=B_1 & \text{on } \Gamma. \end{cases}$$

Then $0\leqslant\phi\in L^\infty(\Omega)$, $\lim_{t\to\infty}|u(t,x,u_0)|\leqslant\phi(x)$, *uniformly in* $x\in\bar\Omega$ *and for* u_0 *in bounded subsets of* $X_0^{\frac12}$. *In particular, for each* $v\in\mathcal{A}_0$ *we have* $|v(x)|\leqslant\phi(x)$.

3. Compact convergence and linear theory

In this section we study the *convergence* of the solutions of the elliptic problems

$$\begin{cases} -\operatorname{div}\big(p_\epsilon(x)\nabla u^\epsilon\big)+\lambda u^\epsilon=h^\epsilon & \text{in } \Omega,\\ p_\epsilon(x)\dfrac{\partial u^\epsilon}{\partial \vec n}=0 & \text{on } \Gamma \end{cases} \tag{3.10}$$

to the solution of the limiting problem

$$\begin{cases} -\operatorname{div}\big(p_0(x)\nabla u\big)+\lambda u=h_0 & \text{in } \Omega_1,\\ \dfrac{1}{|\Omega_{0,i}|}\displaystyle\int_{\Gamma_{0,i}} p_0(x)\frac{\partial u}{\partial \vec n}\,\mathrm{d}x+\int_{\Omega_{0,i}}\lambda u_{\Omega_{0,i}}\,\mathrm{d}x=h_{\Omega_{0,i}} & i=1,\dots,m,\\ p_0(x)\dfrac{\partial u}{\partial \vec n}=0 & \text{on } \Gamma \end{cases} \tag{3.11}$$

when h_ϵ *converges* h_0 in a sense to be specified. That translates into *convergence* of A_ϵ^{-1} to A_0^{-1} which will imply *convergence* of eigenvalues and eigenfunctions of A_ϵ to eigenvalues and eigenfunctions of A_0 and *convergence* of the linear semigroups $e^{-A_\epsilon t}$ to the linear semigroup $e^{-A_0 t}$.

Remark 3.1. Clearly X_0^α is continuously embedded in X_ϵ^α for $\alpha=0$ and $\alpha=\frac12$. As a consequence of that and of the characterization of the fractional power spaces via complex interpolation (see [19, Theorem 1.15.3 and Remark 1.15.2|1]) we obtain that X_0^α is continuously embedded in X_ϵ^α, $0\leqslant\alpha\leqslant\frac12$.

In order to give meaning to the above mentioned convergence we need to introduce some terminology.

Definition 3.1. We say that a sequence $\{u^\epsilon\}$ with $u^\epsilon\in X_\epsilon^{\frac12}$, $X_\epsilon^{\frac12}$*-converges* to $u\in X_0^{\frac12}$ as $\epsilon\to0$ if $\|u^\epsilon-u\|_{X_\epsilon^{\frac12}}\xrightarrow{\epsilon\to0}0$ and we say that $\{u^\epsilon\}$ *converges* $X_\epsilon^{\frac12}$*-weakly* to $u\in X_0^{\frac12}$ as $\epsilon\to0$ if

$$\int_\Omega p_\epsilon(x)\nabla u^\epsilon\nabla\phi\,\mathrm{d}x+\int_\Omega\lambda u^\epsilon\phi\,\mathrm{d}x\to\int_{\Omega_1}p_0(x)\nabla u\nabla\phi\,\mathrm{d}x+\int_\Omega\lambda u\phi\,\mathrm{d}x,\quad \text{for any } \phi\in X_0^{\frac12}.$$

Since $X_0^{\frac12}$ is a closed subspace $X_\epsilon^{\frac12}$, given $h_\epsilon\in X_\epsilon^{-\frac12}$, we have that h_ϵ is a bounded linear functional in $X_\epsilon^{\frac12}$ and its restriction to $X_0^{\frac12}$, also denoted by h_ϵ, is an element of $X_0^{-\frac12}$.

Definition 3.2. Given sequences $\{\epsilon_n\}\to0$, $\{h_{\epsilon_n}\}, h_{\epsilon_n}\in X_{\epsilon_n}^{-\frac12}$, $n\in\mathbb{N}$, and $h_0\in X_0^{-\frac12}$, we say that h_{ϵ_n} *converges to* h_0 and we denote $h_{\epsilon_n}\to h_0$ if $\|h_{\epsilon_n}-h_0\|_{X_0^{-\frac12}}\xrightarrow{n\to\infty}0$ and $h_{\epsilon_n}(u_{\epsilon_n})\to h_0(u)$ whenever u_{ϵ_n} $X_{\epsilon_n}^{\frac12}$-converges to u.

3.1. Compact convergence of resolvents

Many of the results that we use in this article to show the continuity of attractors follow from a special notion of convergence for families of linear operators called compact convergence. Next we introduce this notion adapted to the situation we encounter here.

Definition 3.3. Let X (here $X = \mathrm{L}^2(\Omega)$) be a Banach space and X_0 (here $X_0 = \mathrm{L}^2_{\Omega_0}(\Omega)$) a closed subspace of X.

(i) A family of bounded linear operators $\{S_\epsilon \in \mathcal{L}(X)\colon \epsilon \in (0,1]\}$ is said to be *convergent* to the bounded linear operator $S \in \mathcal{L}(X_0)$ as $\epsilon \to 0$ if $S_\epsilon v_\epsilon \xrightarrow{\epsilon\to 0} Sv$ in X, whenever $X \ni v_\epsilon \xrightarrow{\epsilon\to 0} v \in X_0$ in X, and we denote this convergence by $S_\epsilon \xrightarrow{\epsilon\to 0} S$.

(ii) We say that a family of operators $\{S_\epsilon \in \mathcal{L}(X)\colon \epsilon \in (0,1]\}$ is X_0-*collectively compact* if each operator S_ϵ is compact, $\epsilon \in (0,1]$, and for any bounded sequence $\{v_{\epsilon_n}\}$ in X, $\epsilon_n \xrightarrow{n\to\infty} 0$, there exists a subsequence $\{v_{n_k}\}$ such that $\{S_{\epsilon_{n_k}} v_{\epsilon_{n_k}}\}$ converges to an element of X_0.

(iii) We say that a family of operators $\{S_\epsilon \in \mathcal{L}(X)\colon \epsilon \in (0,1]\}$ *converges X_0-compactly* to $S \in \mathcal{L}(X_0)$ if this family is X_0-collectively compact and $S_\epsilon \xrightarrow{\epsilon\to 0} S$.

We often simply say *collectively compact family* or *compact convergence* since, for our purpose, the subspace X_0 is fixed.

The following result is an immediate consequence of Lemma 6.6 in [11].

Lemma 3.1. *Assume that $S_\epsilon \in \mathcal{L}(X)$, $\epsilon \in (0,\epsilon_0]$, converges compactly to S_0 as $\epsilon \to 0$. If $\mathcal{N}(I+S_0) = \{0\}$ then, there exist an $\bar\epsilon > 0$ and $M > 0$ such that*

$$\left\|(I+S_\epsilon)^{-1}\right\|_{\mathcal{L}(X)} \leqslant M, \quad \epsilon \in [0,\bar\epsilon].$$

Next result ensures that the family of solutions of (3.10) converges to a solution of the limiting problem (3.11) in the sense of Definition 3.1. For the proof we refer to [8] where the result is proved in this form using the results of [17].

Theorem 3.1. *Let $\{\epsilon_n\}$ be a sequence in $(0,\epsilon_0]$ that converges to zero, $\{h_{\epsilon_n}\}$, $h_{\epsilon_n} \in X_{\epsilon_n}^{-\frac12}$, $n \in \mathbb{N}$, such that $\|h_{\epsilon_n}\|_{X_{\epsilon_n}^{-\frac12}} \leqslant 1$ for any $n \in \mathbb{N}$ and $\{u_{\epsilon_n}\}$ satisfying $A_{\epsilon_n} u_{\epsilon_n} = h_{\epsilon_n}$. If $\{h_{\epsilon_n}\}$ converges $X_\epsilon^{-\frac12}$-weakly to $h \in X_0^{-\frac12}$, then there exists a subsequence of $\{u_{\epsilon_n}\}$, that we also denote by $\{u_{\epsilon_n}\}$, and $u \in X_0^{\frac12}$ such that $\{u_{\epsilon_n}\}$ converges to u $X_\epsilon^{\frac12}$-weakly and strongly in $X = \mathrm{L}^2(\Omega)$, where u is the solution of $A_0 u = h$.*

Next we prove the compact convergence of ${A_\epsilon}^{-1}$ to ${A_0}^{-1}$ as $\epsilon \to 0$.

Theorem 3.2. *The family $\{{A_\epsilon}^{-1} \in \mathcal{L}(X)\colon \epsilon \in (0,\epsilon_0]\}$ converges compactly to the operator $A_0^{-1} \in \mathcal{L}(X_0)$ as $\epsilon \to 0$.*

Proof. Let $\{h_\epsilon\} \subset X$ be a sequence that $h_\epsilon \to h \in X_0$ in X. Then ${A_\epsilon}^{-1} h_\epsilon := u^\epsilon \in X_\epsilon^{\frac12}$ and it follows from Theorem 3.1 that there exists $u \in X_0^{\frac12}$ such that $\{u^\epsilon\}$ converges to u $X_\epsilon^{\frac12}$-weakly and strongly in X, that is, ${A_\epsilon}^{-1} \xrightarrow{\epsilon\to 0} A_0^{-1}$.

If $\{h_\epsilon\}$ a bounded family in X and v^ϵ is such that $A_\epsilon v^\epsilon = h_\epsilon$, we have

$$\int_\Omega p_\epsilon(x)\left|\nabla v^\epsilon\right|^2 \mathrm{d}x + \int_\Omega \lambda\left|v^\epsilon\right|^2 \mathrm{d}x = \left\langle h_\epsilon, v^\epsilon\right\rangle_{-1,1} \leqslant \|h_\epsilon\|_{X_\epsilon^{-\frac12}} \left\|v^\epsilon\right\|_{X_\epsilon^{\frac12}},$$

and $\{v^\epsilon\}_{X_\epsilon^{\frac12}}$ is bounded and has a convergent subsequence in X for $u \in X_0^{\frac12}$. $\square$

Remark 3.2. If $A_\epsilon^{-1} \xrightarrow{CC} A_0^{-1}$, it follows from Lemma 6.12 in [11] that $A_\epsilon^{-\theta} \xrightarrow{CC} A_0^{-\theta}$, $0 < \theta < 1$.

3.2. Spectral convergence

In this section we study the spectral properties of the operators A_ϵ, $\epsilon \in [0,\epsilon_0]$. The fact that the spectrum of A_ϵ, $\epsilon \in (0,\epsilon_0]$, is close to the spectrum of A_0 comes from the compact convergence as an abstract result, so we will omit its proof, which can be found in [4,8,11]. For an alternative proof the reader can also consult [17].

The $X_\epsilon^{\frac12}$-convergence of the resolvent operators is an important tool to show the convergence of linear and nonlinear semigroups associated to the problems (2.9). We recall that, for $0 \leqslant \alpha \leqslant \frac12$, $X_0^\alpha \subset X_\epsilon^\alpha$ (see Remark 3.1).

Also recall that, if C_ϵ is a sectorial operator with $\|\mathrm{e}^{-C_\epsilon t}\| \leqslant \tilde M$ and $\tilde M$ independent of ϵ, it follows from the proof of Theorem 1.4.4 in [15], the constant M appearing in the Moment Inequality; that is,

$$\left\|C_\epsilon^\alpha x\right\|_X \leqslant M\left\|C_\epsilon^\gamma x\right\|_X^{\frac{\alpha-\beta}{\gamma-\beta}} \left\|C_\epsilon^\beta x\right\|_X^{\frac{\gamma-\alpha}{\gamma-\beta}}, \tag{3.12}$$

depends exclusively on the bound $\tilde M$ for the semigroup for $\beta < \alpha < \gamma$. Hence M can be chosen uniform with respect to ϵ.

Lemma 3.2. *Let K be a compact subset of $\rho(-A_0)$. Then there exists a constant $\epsilon_K > 0$ such that $K \subset \rho(-A_\epsilon)$ for every $\epsilon \in (0, \epsilon_K]$ and*

$$\sup_{\alpha\in[0,1]} \sup_{\epsilon\in(0,\epsilon_K]} \sup_{\mu\in K} \left\| \left(\mu + A_\epsilon\right)^{-1} \right\|_{\mathcal{L}(X, X_\epsilon^\alpha)} < \infty. \tag{3.13}$$

If $f_\Omega \in \mathrm{L}^2(\Omega)$, $g_\Gamma \in \mathrm{L}^2(\Gamma)$ and $h = f_\Omega + g_\Gamma \in X_0^{-\frac{s}{2}}$, $\frac{1}{2} \leqslant s < 1$, is defined by

$$\langle h, \phi\rangle_{-s,s} = \int_\Omega f_\Omega \phi \,\mathrm{d}x + \int_\Gamma g_\Gamma \phi \,\mathrm{d}x, \quad \forall \phi \in X_0^{\frac{s}{2}}, \tag{3.14}$$

then we extend $h : X^{\frac{s}{2}} \to \mathbb{R}$ to $X_\epsilon^{\frac{s}{2}}$, using (3.14), and

$$\sup_{\mu\in K} \left\| (\mu + A_\epsilon)^{-1} h - (\mu + A_0)^{-1} h \right\|_{X_\epsilon^{\frac{1}{2}}} \xrightarrow{\epsilon\to 0} 0. \tag{3.15}$$

Furthermore, if J is a subset of $X_0^{-\frac{s}{2}}$ constituted by elements of the form $f_\Omega + g_\Gamma$, with f_Ω (g_Γ, respectively) varying in a bounded subset of $\mathrm{L}^2(\Omega)$ ($\mathrm{L}^2(\Gamma)$ respectively), we have

$$\sup_{\substack{\mu\in K\\ h\in J}} \left\| (\mu + A_\epsilon)^{-1} h - (\mu + A_0)^{-1} h \right\|_{X_\epsilon^{\frac{1}{2}}} \xrightarrow{\epsilon\to 0} 0. \tag{3.16}$$

Proof. It follows from the proof of Lemma 4.9 in [4] that there is $\epsilon_K > 0$ such that $K \subset \rho(-A_\epsilon)$ for any $\epsilon \in (0, \epsilon_K]$. The proof of (3.13) with $\alpha = 0$ follows from the proof of Lemma 4.11 in [4]. Now, using that $A_\epsilon(\mu + A_\epsilon)^{-1} = I - \mu(\mu + A_\epsilon)^{-1}$ and the Moment Inequality, the result follows.

Let $u^\epsilon = (\mu + A_\epsilon)^{-1} h$ and $u = (\mu + A_0)^{-1} h$. Then

$$\begin{aligned}
\left\langle h, u^\epsilon - u\right\rangle_{-s,s} &= \int_\Omega p_\epsilon(x)\left|\nabla u^\epsilon\right|^2 \mathrm{d}x - \int_{\Omega_1} p_0(x)|\nabla u|^2 \,\mathrm{d}x + (\lambda+\mu)\int_\Omega \left(u^\epsilon\right)^2 \mathrm{d}x - (\lambda+\mu)\int_\Omega u^2 \,\mathrm{d}x \\
&= \int_\Omega p_\epsilon(x)\left|\nabla u^\epsilon - \nabla u\right|^2 \mathrm{d}x + (\lambda+\mu)\int_\Omega \left(u^\epsilon - u\right)^2 \mathrm{d}x - \int_{\Omega_1} \left(p_\epsilon(x) - p_0(x)\right)|\nabla u|^2 \,\mathrm{d}x \\
&\quad + 2\int_{\Omega_1} p_\epsilon(x)\nabla u^\epsilon \nabla u \,\mathrm{d}x + 2(\lambda+\mu)\int_\Omega u^\epsilon u \,\mathrm{d}x - 2\int_{\Omega_1} p_0(x)|\nabla u|^2 \,\mathrm{d}x - 2(\lambda+\mu)\int_\Omega u^2 \,\mathrm{d}x \\
&= \int_\Omega p_\epsilon(x)\left|\nabla u^\epsilon - \nabla u\right|^2 \mathrm{d}x + (\lambda+\mu)\int_\Omega \left(u^\epsilon - u\right)^2 \mathrm{d}x - \int_{\Omega_1} \left(p_\epsilon(x) - p_0(x)\right)|\nabla u|^2 \,\mathrm{d}x.
\end{aligned}$$

It follows from Theorem 3.1 that $u^\epsilon \to u$ $X_\epsilon^{\frac{1}{2}}$-weakly. Thus, $\langle h, u^\epsilon - u\rangle_{-s,s} \to 0$. Furthermore, since $p_\epsilon \to p_0$ uniformly in Ω_1 and $\int_{\Omega_1} |\nabla u|^2 \,\mathrm{d}x < \infty$, from expression above we conclude that u^ϵ $X_\epsilon^{\frac{1}{2}}$-converges to u.

Now consider J a bounded subset of $X_0^{-\frac{s}{2}}$ in the following manner: there are M_1 and M_2 such that if $h \in J$, then

$$h = f_\Omega + g_\Gamma$$

with $\|f_\Omega\|_{\mathrm{L}^2(\Omega)} \leqslant M_1$ and $\|g_\Gamma\|_{\mathrm{L}^2(\Gamma)} \leqslant M_2$. Denote by J_ϵ the set of extensions $h_\epsilon : X_\epsilon^{\frac{s}{2}} \to \mathbb{R}$ of elements $h : X_0^{\frac{s}{2}} \to \mathbb{R}$ of J to $X_\epsilon^{\frac{s}{2}}$, using (3.14). It is easy to check that

$$\sup_{\epsilon\in(0,\epsilon_0]} \sup_{h_\epsilon\in J_\epsilon} \|h_\epsilon\|_{X_\epsilon^{-\frac{s}{2}}} < \infty.$$

It results from $\sup_{\epsilon\in(0,\epsilon_0]} \|u^\epsilon - u\|_{X_\epsilon^{\frac{1}{2}}} < \infty$, $\|u^\epsilon - u\|_X \to 0$ as $\epsilon \to 0$ and from (3.12) that $\|u^\epsilon - u\|_{X_\epsilon^{\frac{s}{2}}} \to 0$ as $\epsilon \to 0$. This guarantees that $\sup_{h\in J} \langle h, u^\epsilon - u\rangle_{-s,s} \xrightarrow{\epsilon\to 0} 0$. The result now follows from

$$\left\langle h, u^\epsilon - u\right\rangle_{-s,s} = \int_\Omega p_\epsilon(x)\left|\nabla u^\epsilon - \nabla u\right|^2 \mathrm{d}x + (\lambda+\mu)\int_\Omega \left(u^\epsilon - u\right)^2 \mathrm{d}x - \int_{\Omega_1} \left(p_\epsilon(x) - p_0(x)\right)|\nabla u|^2 \,\mathrm{d}x. \qquad \square$$

Let μ be an isolated point of $\sigma(-A_0)$ and $\delta > 0$ satisfying $\{z \in \mathbb{C}\colon |z - \mu| \leqslant \delta\} \cap \sigma(-A_0) = \{\mu\}$. Since $K = \{z \in \mathbb{C}\colon |z - \mu| = \delta\}$ is compact and $K \subset \rho(-A_0)$, it follows from the previous Lemma 3.2 that there exists ϵ_K such that $K \subset \rho(-A_\epsilon)$, for $\epsilon \in [0, \epsilon_K]$. For $\epsilon \in [0, \epsilon_K]$, we associate its generalized eigenspace $W(\mu, -A_\epsilon) = Q_\epsilon(\mu, -A_\epsilon)X$ where

$$Q_\epsilon(\mu, -A_\epsilon) = \frac{1}{2\pi i} \int_{|\xi-\mu|=\delta} (\xi I + A_\epsilon)^{-1} \,\mathrm{d}\xi.$$

The following result is now a consequence of Theorem 4.10 in [4].

Theorem 3.3. *The following statements hold*:

(i) *For any* $\mu_0 \in \sigma(-A_0)$, *there are sequences* $\epsilon_n \to 0$ *and* $\{\mu_n\}$, $\mu_n \in \sigma(-A_{\epsilon_n})$, $n \in \mathbb{N}$, *such that* $\mu_n \to \mu_0$ *as* $n \to \infty$.
(ii) *If for some sequence* $\epsilon_n \to 0$, *one has* $\mu_n \to \mu_0$ *as* $n \to \infty$, *with* $\mu_n \in \sigma(-A_{\epsilon_n})$, $n \in \mathbb{N}$, *then* $\mu_0 \in \sigma(-A_0)$.
(iii) *There is* $\epsilon_0 > 0$ *such that* $\dim W(\mu, -A_\epsilon) = \dim W(\mu, -A_0)$, *for all* $0 < \epsilon \leqslant \epsilon_0$.
(iv) *For any* $u \in W(\mu_0, -A_0)$, *there is a sequence* $\{u^\epsilon\}$, $u^\epsilon \in W(\mu_0, -A_\epsilon)$, *such that* $u^\epsilon \xrightarrow{\epsilon \to 0} u$ (*here convergence is in* X).
(v) *If* $\epsilon_n \to 0$, *any sequence* $\{u_n\}$, $u_n \in W(\mu, -A_{\epsilon_n})$, $n \in \mathbb{N}$, *with* $\|u_n\|_X = 1$, *has a convergent* (*in* X) *subsequence and any limit point of this sequence belongs to* $W(\mu_0, -A_0)$.

3.3. Convergence of linear semigroups

The continuity of resolvent operators allows us to obtain the continuity of linear semigroups, that will be of fundamental importance in the analysis of the nonlinear dynamics.

The properties of the operators $A_\epsilon, \epsilon \in [0, \epsilon_0]$, stated in (2.8) imply the existence of a constant M_ω, independent of ϵ, such that

$$\left\| e^{-A_\epsilon t} u \right\|_{X_\epsilon^{\frac{1}{2}}} \leqslant M_\omega t^{-(\frac{1+s}{2})} e^{-\omega t} \|u\|_{X_\epsilon^{-\frac{s}{2}}}, \tag{3.17}$$

for all $t > 0$ and for all $\epsilon \in [0, \epsilon_0]$, $s \in [\frac{1}{2}, 1)$. We use these estimates and the resolvent convergence obtained in Lemma 3.2 to show that linear semigroups associated to these operators behave continuously at $\epsilon \to 0$.

Theorem 3.4. *Let* K *be a compact subset of* $X_0^{-\frac{s}{2}}$, $\frac{1}{2} < s < 1$ *and* $0 < \theta < \frac{1}{2}$. *If* $h = f_\Omega + g_\Gamma \in K$ *is like in* (3.14) *and we extend* h, *using* (3.14), *to view* h *as an element of* $X_\epsilon^{-\frac{s}{2}}$, *then there exists a function* $\nu : (0, \epsilon_0] \to \mathbb{R}^+$, $\nu(\epsilon) \xrightarrow{\epsilon \to 0} 0$ *such that*

$$\sup_{h \in K} \left\| e^{-A_\epsilon t} h - e^{-A_0 t} h \right\|_{X_\epsilon^{\frac{1}{2}}} \leqslant \nu(\epsilon) e^{-\omega t} t^{-\theta - \frac{(1+s)}{2}}, \quad \textit{for all } t > 0.$$

Furthermore, if K_1 *is a compact subset* $X_0^{\frac{1}{2}}$ *then there is a function* $\tilde{\nu} : (0, \epsilon_0] \to \mathbb{R}^+$, $\tilde{\nu}(\epsilon) \xrightarrow{\epsilon \to 0} 0$ *such that*

$$\sup_{u \in K_1} \left\| e^{-A_\epsilon t} u - e^{-A_0 t} u \right\|_{X_\epsilon^{\frac{1}{2}}} \leqslant \tilde{\nu}(\epsilon) e^{-\omega t} t^{-\theta - \frac{1}{2}}, \quad \textit{for all } t > 0.$$

Proof. Let us prove the first inequality. The second one can be proved in a similar way. We will separate the estimate for $t \in (0, \delta)$ and $t \geqslant \delta$, with $\delta \in (0, 1)$ fixed.

For $0 < t < \delta$, we have, from (3.17),

$$\left\| e^{-A_\epsilon t} h - e^{-A_0 t} h \right\|_{X_\epsilon^{\frac{1}{2}}} \leqslant 2 M_\omega e^{-\omega t} \delta^\theta t^{-\theta - (\frac{1+s}{2})}$$

since h is in a compact subset of $X_0^{-\frac{s}{2}}$. Also, since the operators A_ϵ, $\epsilon \in [0, \epsilon_0]$, are sectorial, we have for $\bar{\omega} < \lambda$,

$$e^{(-A_\epsilon + \bar{\omega} I)t} = \frac{1}{2\pi i} \int_{\tilde{\Gamma}} e^{(\mu + \bar{\omega})t} (\mu + \bar{\omega} + A_\epsilon - \bar{\omega})^{-1} \, d\mu, \quad \epsilon \in [0, \epsilon_0],$$

where $\tilde{\Gamma}$ is the boundary of $\Sigma_{-\lambda,\phi} = \{\mu \in \mathbb{C}\colon |\arg(\mu + \lambda)| \leqslant \phi\}$, with $\frac{\pi}{2} > \phi > \pi$ oriented in such a way that the imaginary part of μ increases as μ runs $\tilde{\Gamma}$.

After changing variables $\mu + \bar{\omega} \mapsto \mu$ and denoting $B_\epsilon := A_\epsilon - \bar{\omega} I$, $\epsilon \in [0, \epsilon_0]$, our aim is to estimate, for $t \geqslant \delta$, the difference

$$2\pi t^{\frac{1+s}{2}} \left\| e^{-B_\epsilon t} h - e^{-B_0 t} h \right\|_{X_\epsilon^{\frac{1}{2}}} = \left\| \int_{\Gamma_0} t^{\frac{1+s}{2}} e^{\mu t} \left[(\mu + B_\epsilon)^{-1} h - (\mu + B_0)^{-1} h \right] d\mu \right\|_{X_\epsilon^{\frac{1}{2}}}, \tag{3.18}$$

where Γ_0 the boundary of $\Sigma_{0,\phi}$.

To see that the integral on the right-hand side in (3.18) is convergent, notice that, from the fact that B_ϵ is self-adjoint and positive, for $\mu \in \Gamma_0$, $\|(\mu + B_\epsilon)^{-1}\|_{\mathcal{L}(X_\epsilon^{-\frac{s}{2}})} \leqslant \frac{M}{|\mu|}$, for all $\epsilon \in [0, \epsilon_0]$. Also $\|(\mu + B_\epsilon)^{-1}\|_{\mathcal{L}(X_\epsilon^{-\frac{s}{2}})}$ is bounded in Γ_0. Thus,

$$\begin{aligned} \left\| B_\epsilon (\mu + B_\epsilon)^{-1} h \right\|_{X_\epsilon^{-\frac{s}{2}}} &= \left\| \left(I - \mu (\mu + B_\epsilon)^{-1} \right) h \right\|_{X_\epsilon^{-\frac{s}{2}}} \\ &\leqslant \|h\|_{X_\epsilon^{-\frac{s}{2}}} + |\mu| \left\| (\mu + B_\epsilon)^{-1} h \right\|_{X_\epsilon^{-\frac{s}{2}}} \\ &\leqslant M_1 \|h\|_{X_\epsilon^{-\frac{s}{2}}}. \end{aligned}$$

From (3.12) for B_ϵ with powers $\beta = \frac{-s}{2} < \alpha = \frac{1}{2} < \gamma = 1 - \frac{s}{2}$, we obtain

$$\begin{aligned}\left\|(\mu + B_\epsilon)^{-1}h\right\|_{X_\epsilon^{\frac{1}{2}}} &= \left\|B_\epsilon^{\frac{1}{2}}(\mu + B_\epsilon)^{-1}h\right\|_X \\ &\leqslant C\left\|B_\epsilon^{(1-\frac{s}{2})}(\mu + B_\epsilon)^{-1}h\right\|_X^{\frac{1+s}{2}}\left\|B_\epsilon^{-\frac{s}{2}}(\mu + B_\epsilon)^{-1}h\right\|_X^{\frac{1-s}{2}} \\ &= C\left\|B_\epsilon(\mu + B_\epsilon)^{-1}h\right\|_{X_\epsilon^{-\frac{s}{2}}}^{\frac{1+s}{2}}\left\|(\mu + B_\epsilon)^{-1}h\right\|_{X_\epsilon^{-\frac{s}{2}}}^{\frac{1-s}{2}} \\ &\leqslant CM_1^{\frac{1+s}{2}}\|h\|_{X_\epsilon^{-\frac{s}{2}}}^{\frac{1+s}{2}}\frac{M^{\frac{1-s}{2}}}{|\mu|^{\frac{1-s}{2}}}\|h\|_{X_\epsilon^{-\frac{s}{2}}}^{\frac{1-s}{2}} \\ &\leqslant \frac{M_2}{|\mu|^{\frac{1-s}{2}}}\|h\|_{X_\epsilon^{-\frac{s}{2}}},\end{aligned}$$

and so

$$\left\|(\mu + B_\epsilon)^{-1}h - (\mu + B_0)^{-1}h\right\|_{X_\epsilon^{\frac{1}{2}}} \leqslant \frac{2M_2}{|\mu|^{\frac{1-s}{2}}}\|h\|_{X_\epsilon^{-\frac{s}{2}}}$$

and also $\|(\mu + B_\epsilon)^{-1} - (\mu + B_0)^{-1}\|_{\mathcal{L}(X_\epsilon^{-\frac{s}{2}}, X_\epsilon^{\frac{1}{2}})}$ is bounded uniformly in Γ_0.

Therefore, for the integral mentioned we have

$$\begin{aligned}\left\|\int_{\Gamma_0} t^{\frac{1+s}{2}}e^{\mu t}\left[(\mu + B_\epsilon)^{-1}h - (\mu + B_0)^{-1}h\right]d\mu\right\|_{X_\epsilon^{\frac{1}{2}}} &\leqslant t^{\frac{1+s}{2}}\int_{\Gamma_0}\left|e^{\mu t}\right|\frac{2M_2}{|\mu|^{\frac{1-s}{2}}}\|h\|_{X_\epsilon^{-\frac{s}{2}}}\,d|\mu| \\ &\overset{\gamma=\mu t}{=} 2M_2 t^{\frac{1+s}{2}}\int_{\Gamma_0}\left|e^{\gamma}\right||\gamma|^{\frac{s-1}{2}}t^{\frac{1-s}{2}}\|h\|_{X_\epsilon^{-\frac{s}{2}}}\frac{d|\gamma|}{t} \\ &= M_3\|h\|_{X_\epsilon^{-\frac{s}{2}}}.\end{aligned}$$

Let us now deal with $t \geqslant \delta$. Given $\eta > 0$, we write $\Gamma_0 = \Gamma_1^\eta \cup \Gamma_2^\eta$, with Γ_1^η bounded and Γ_2^η satisfying (notice that $\|(\mu + B_\epsilon)^{-1} - (\mu + B_0)^{-1}\|_{\mathcal{L}(X_\epsilon^{-\frac{s}{2}}, X_\epsilon^{\frac{1}{2}})}$ is bounded uniformly in Γ_0)

$$\sup_{\epsilon\in(0,\epsilon_0]}\sup_{h\in K}\left\|t^{\frac{1+s}{2}}\int_{\Gamma_2^\eta}e^{\mu t}\left[(\mu + B_\epsilon)^{-1}h - (\mu + B_0)^{-1}h\right]d\mu\right\|_{X_\epsilon^{\frac{1}{2}}} < \frac{\eta}{2}.$$

On Γ_1^η, we change variables $\beta = \mu t$ in (3.18) and obtain

$$\left\|t^{\frac{1+s}{2}}\int_{\Gamma_1^\eta}e^{\beta}\left[\left(\beta t^{-1} + B_\epsilon\right)^{-1}h - \left(\beta t^{-1} + B_0\right)^{-1}h\right]\frac{d\beta}{t}\right\|_{X_\epsilon^{\frac{1}{2}}}.$$

Again using (3.12) and the uniform estimate $\|(\mu + B_\epsilon)^{-1}\|_{\mathcal{L}(X_\epsilon^{-\frac{s}{2}})} \leqslant \frac{M}{|\mu|}$, it follows that

$$\left\|\left(\beta t^{-1} + B_\epsilon\right)^{-1}h - \left(\beta t^{-1} + B_0\right)^{-1}h\right\|_{X_\epsilon^{\frac{1}{2}}} \leqslant 2\frac{M^{\frac{1+s}{2}}}{|\beta t^{-1}|^{\frac{1+s}{2}}}\|h\|_X = \frac{M_4}{|\beta|^{\frac{1+s}{2}}}t^{\frac{1+s}{2}}\|h\|_X,$$

that is,

$$\sup_{h\in K}\left\|\left(\beta t^{-1} + B_\epsilon\right)^{-1}h - \left(\beta t^{-1} + B_0\right)^{-1}h\right\|_{X_\epsilon^{\frac{1}{2}}} \leqslant \frac{M_5}{|\beta|^{\frac{1+s}{2}}}t^{\frac{1+s}{2}}.$$

Also, since $t \geqslant \delta$ we have $t^{-1} \in [0, \delta^{-1}] \Subset \mathbb{R}$ and, for $\beta \in \Gamma_1^\eta$, βt^{-1} is in a compact subset of $\rho(-B_0)$. Hence, just like in Lemma 3.2, we guarantee the existence of a function $\tilde{\nu} : (0, \epsilon_0] \to \mathbb{R}^+$, with $\tilde{\nu}(\epsilon) \to 0$ as $\epsilon \to 0$, with

$$\sup_{h\in K}\left\|\left(\beta t^{-1} + B_\epsilon\right)^{-1}h - \left(\beta t^{-1} + B_0\right)^{-1}h\right\|_{X_\epsilon^{\frac{1}{2}}} \leqslant \tilde{\nu}(\epsilon), \quad \text{for } t \geqslant \delta \text{ and } \beta \in \Gamma_1^\eta.$$

Now, for $\theta \in (0, \frac{1}{2})$ fixed, we interpolate the last two expressions, obtaining

$$\sup_{h\in K}\left\|\left(\beta t^{-1} + B_\epsilon\right)^{-1}h - \left(\beta t^{-1} + B_0\right)^{-1}h\right\|_{X_\epsilon^{\frac{1}{2}}} \leqslant \frac{M_5^{1-2\theta}}{|\beta|^{(\frac{1+s}{2})-\theta}}t^{(\frac{1+s}{2})-\theta}\tilde{\nu}(\epsilon)^{2\theta}.$$

Thus, there exists $\bar{\epsilon} > 0$ such that for $\epsilon \in (0, \bar{\epsilon}]$, we have

$$\sup_{h\in K}\left\| 2\pi t^{\theta-(\frac{1+s}{2})} \int_{\Gamma_1^\eta} e^{\beta}\left[\left(\beta t^{-1}+B_\epsilon\right)^{-1}h-\left(\beta t^{-1}+B_0\right)^{-1}h\right] d\beta \right\|_{X_\epsilon^{\frac{1}{2}}} \leqslant \int_{\Gamma_1^\eta} \left|e^{\beta}\right| \frac{M_6}{|\beta|^{(\frac{1+s}{2})-\theta}} \tilde{\nu}(\epsilon)^{2\theta}\, d|\beta| < \frac{\eta}{2},$$

that is,

$$\sup_{h\in K}\left\| e^{-B_\epsilon t}h - e^{-B_0 t}h\right\|_{X_\epsilon^{\frac{1}{2}}} \leqslant \frac{\eta}{4\pi} t^{-\theta-(\frac{1+s}{2})} + \frac{\eta}{4\pi} t^{-\theta-(\frac{1+s}{2})}, \quad \text{for } t \geqslant \delta.$$

Therefore, there exists a function $\nu : (0, \epsilon_0] \to \mathbb{R}^+$, with $\nu(\epsilon) \xrightarrow{\epsilon\to 0} 0$, satisfying

$$\sup_{h\in K}\left\| e^{-A_\epsilon t}h - e^{-A_0 t}h\right\|_{X_\epsilon^{\frac{1}{2}}} \leqslant \nu(\epsilon) e^{-\omega t} t^{-(\frac{1+s}{2})-\theta}, \quad \text{for all } t > 0. \qquad \square$$

4. Continuity of attractors

In this section we study the continuity of the family of attractors $\{\mathcal{A}_\epsilon\colon\ 0 \in (0, \epsilon_0]\}$ for (1.1) and $\mathcal{A}_0$ for (1.2) as $\epsilon \to 0$. Let us come back for a moment to the question of existence of attractor for the semigroups associated to problem (1.1). Notice that the problem (1.4) is the limiting problem for

$$\begin{cases} -\operatorname{div}\left(p_\epsilon(x)\nabla u^\epsilon\right) + (\lambda + C_0)u^\epsilon = \mu u^\epsilon & \text{in } \Omega,\\ p_\epsilon(x)\dfrac{\partial u^\epsilon}{\partial \vec{n}} + B_0 u^\epsilon = 0 & \text{on } \Gamma. \end{cases} \tag{4.1}$$

In order to obtain the existence of a global attractor for (1.1) we prove the compact convergence of operators of the form $(A_\epsilon + D_\epsilon)^{-1}$, with the additional assumption that $A_\epsilon^{-\frac{s}{2}} D_\epsilon A_\epsilon^{-\frac{\eta}{2}}$ converges to $A_0^{-\frac{s}{2}} D_0 A_0^{-\frac{\eta}{2}}$ with $s, \eta \in (0, 1)$.

Lemma 4.1. *Assume that $A_\epsilon^{-1} \xrightarrow{CC} A_0^{-1}$ and that $D_\epsilon \in \mathcal{L}(X_\epsilon^{\frac{\eta}{2}}, X_\epsilon^{-\frac{s}{2}})$ and $A_\epsilon^{-\frac{s}{2}} D_\epsilon A_\epsilon^{-\frac{\eta}{2}} \to A_0^{-\frac{s}{2}} D_0 A_0^{-\frac{\eta}{2}}$ as $\epsilon \to 0$, with $0 < s, \eta < 1$. Then $(A_\epsilon + D_\epsilon)^{-1}$ converges compactly to $(A_0 + D_0)^{-1}$ as $\epsilon \to 0$.*

Proof. Notice that $(A_\epsilon + D_\epsilon)^{-1} = A_\epsilon^{-\frac{1}{2}}(I + A_\epsilon^{-\frac{1}{2}} D_\epsilon A_\epsilon^{-\frac{1}{2}})^{-1} A_\epsilon^{-\frac{1}{2}}$. We first deal with $A_\epsilon^{-\frac{1}{2}} D_\epsilon A_\epsilon^{-\frac{1}{2}}$. For $0 < s, \eta < 1$, we may write

$$A_\epsilon^{-\frac{1}{2}} D_\epsilon A_\epsilon^{-\frac{1}{2}} = A_\epsilon^{-\frac{(1-s)}{2}} A_\epsilon^{-\frac{s}{2}} D_\epsilon A_\epsilon^{-\frac{\eta}{2}} A_\epsilon^{-\frac{(1-\eta)}{2}}.$$

Let $\{u_\epsilon\} \subset X$ with $\|u_\epsilon\|_X \leqslant 1$. Since $A_\epsilon^{-\frac{(1-\eta)}{2}}$ is collectively compact (see Remark 3.2), it follows that $A_\epsilon^{-\frac{(1-\eta)}{2}} u^\epsilon$ has a convergent subsequence in X. Now, from the convergence of $A_\epsilon^{-\frac{s}{2}} D_\epsilon A_\epsilon^{-\frac{\eta}{2}}$ and of $A_\epsilon^{-\frac{(1-s)}{2}}$, in the sense of Definition 3.3(i), we conclude that $A_\epsilon^{-\frac{1}{2}} D_\epsilon A_\epsilon^{-\frac{1}{2}} u_\epsilon$ has a convergent subsequence and $A_\epsilon^{-\frac{1}{2}} D_\epsilon A_\epsilon^{-\frac{1}{2}}$ is collectively compact. Moreover, if $u_\epsilon \to u$, it follows easily that $A_\epsilon^{-\frac{1}{2}} D_\epsilon A_\epsilon^{-\frac{1}{2}} \to A_0^{-\frac{1}{2}} D_0 A_0^{-\frac{1}{2}}$.

To prove the compact convergence of $(A_\epsilon + D_\epsilon)^{-1}$, suppose that $X \ni u_\epsilon \to u \in X_0$ in X and notice that, from Remark 3.2, we have

$$w_\epsilon := A_\epsilon^{-\frac{1}{2}} u_\epsilon \to A_0^{-\frac{1}{2}} u =: w.$$

Now if we denote $v_\epsilon := (I + A_\epsilon^{-\frac{1}{2}} D_\epsilon A_\epsilon^{-\frac{1}{2}})^{-1} w_\epsilon$, we have, from Lemma 3.1, that v_ϵ is bounded. Since

$$v_\epsilon + A_\epsilon^{-\frac{1}{2}} D_\epsilon A_\epsilon^{-\frac{1}{2}} v_\epsilon = w_\epsilon$$

it follows that v_ϵ has a convergent subsequence. Furthermore, if $u^\epsilon \to u$ in X, then $v^\epsilon \to v$ along subsequences and

$$v = \left(I + A_0^{-\frac{1}{2}} D_0 A_0^{-\frac{1}{2}}\right) w.$$

Since v is independent of the subsequence taken, we get the convergence. Finally, it follows from Remark 3.2 that $(A_\epsilon + D_\epsilon)^{-1}$ converges compactly for $(A_0 + D_0)^{-1}$. $\square$

Next we intend to obtain a dissipativeness condition for (1.1) as a consequence of dissipativeness condition $(\mathbf{D}_0)$ for (1.2). With this in mind, we will study the compact convergence of the resolvent operators associated to the family of problems (4.1). So we define, for $\epsilon \in [0, \epsilon_0]$

$$D_\epsilon : X_\epsilon^{\frac{\eta}{2}} \to X_\epsilon^{-\frac{s}{2}}$$

by $D_\epsilon u(\varphi) = \int_\Omega C_0 u\varphi \, dx - \int_\Gamma B_0 \gamma(u)\gamma(\varphi)\, dx$ where C_0 and B_0 are the constants appearing in $(\mathbf{D}_0)$. Notice that

$$\|D_\epsilon u^\epsilon - D_0 u\|_{X_\epsilon^{-\frac{s}{2}}} \leqslant \sup_{\substack{\phi \in X_\epsilon^{\frac{s}{2}} \\ \|\phi\| \leqslant 1}} \int_\Omega \big|C_0[u^\epsilon - u]\phi\big| + \sup_{\substack{\phi \in X_\epsilon^{\frac{s}{2}} \\ \|\phi\| \leqslant 1}} \int_\Gamma \big|B_0\big[\gamma\big(u^\epsilon - u\big)\big]\gamma(\phi)\big| \leqslant C\|u^\epsilon - u\|_{X_\epsilon^{\frac{\eta}{2}}}.$$

Therefore, from Lemma 4.1, we have an analogous result to Theorem 3.3 for $A_\epsilon + D\epsilon$ instead of A_ϵ. This and hypothesis $(\mathbf{D}_0)$ ensure that the first eigenvalue of the problem (4.1) is also positive and bounded away from zero, for every $\epsilon \in (0, \bar{\epsilon}]$.

With this, the following result holds (see [6]).

Theorem 4.1. *Let* $(\mathbf{G})$, $(\mathbf{S})$ *and* $(\mathbf{D}_0)$ *be satisfied. Then the semigroup* $\{T_\epsilon(t)\colon t \geqslant 0\}$ *associated to* (1.1) *has a global attractor* $\mathcal{A}_\epsilon$ *in* $X_\epsilon^{\frac{1}{2}}$, $\epsilon \in [0, \bar{\epsilon}]$. *In addition*

$$\sup_{\epsilon \in [0,\epsilon_0]} \sup_{w \in \mathcal{A}_\epsilon} \|w\|_{X_\epsilon^{\frac{1}{2}}} < \infty$$

and

$$\sup_{\epsilon \in [0,\epsilon_0]} \sup_{w \in \mathcal{A}_\epsilon} \|w\|_{L^\infty(\Omega)} < \infty.$$

After obtaining these bounds on the attractor, which are independent of the parameter ϵ, we can change the nonlinearities, without affecting the attractor. Hereafter we assume that the nonlinearities f and g are globally Lipschitz and bounded with bounded derivatives up to second order. More precisely we will assume that there are constants L_f, L_g, C_f and C_g such that

$$\big|j(s) - j(t)\big| \leqslant L_j|t - s|, \quad \text{for any } s, t \in \mathbb{R},\ j = f, g,$$
$$\big|j^{(i)}(s)\big| \leqslant C_j, \quad \text{for any } s \in \mathbb{R},\ j = f, g,\ i = 0, 1, 2.$$

Now, the continuity of nonlinear semigroups follows using the continuity of linear semigroups and Variation of Constants Formula. Notice that both estimates in Theorem 3.4 are needed since

$$\begin{aligned}\big\|T_\epsilon\big(t, u^\epsilon\big) - T_0(t, u)\big\|_{X_\epsilon^{\frac{1}{2}}} \leqslant{}& \big\|e^{-A_\epsilon t}u^\epsilon - e^{-A_0 t}u\big\|_{X_\epsilon^{\frac{1}{2}}} + \int_0^t \big\|\big[e^{-A_\epsilon(t-r)} - e^{-A_0(t-r)}\big]h\big(T_0\big(r, u^\epsilon\big)\big)\big\|_{X_\epsilon^{\frac{1}{2}}} \, dr \\ &+ \int_0^t \big\|e^{-A_\epsilon(t-r)}\big[h\big(T_\epsilon\big(r, u^\epsilon\big)\big) - h\big(T_0(r, u)\big)\big]\big\|_{X_\epsilon^{\frac{1}{2}}} \, dr\end{aligned}$$

and we need to extend $h(T_0(t, u)) : X_0^{\frac{s}{2}} \to \mathbb{R}$ as before to $h(T_0(t, u)) : X_\epsilon^{\frac{s}{2}} \to \mathbb{R}$. With this and Gronwall's Inequality (see [15, §7.1]), proceeding as in [3, Proposition 3.1], we have the following result:

Theorem 4.2. *If* $u \in X_0^{\frac{1}{2}}$ *and* u^ϵ $X_\epsilon^{\frac{1}{2}}$*-converges* u, *then there is a function* $\bar{\nu} : (0, \infty) \times (0, \epsilon_0] \to \mathbb{R}^+$, *with* $\bar{\nu}(t, \epsilon) \xrightarrow{\epsilon \to 0} 0$, *uniformly for* t *in compact subsets of* $(0, \infty)$, *satisfying*

$$\big\|T_\epsilon\big(t, u^\epsilon\big) - T_0(t, u)\big\|_{X_\epsilon^{\frac{1}{2}}} \leqslant \bar{\nu}(t, \epsilon), \quad t \in (0, \tau).$$

To prove upper and lower semicontinuity of attractors we use the following auxiliar lemma, which proof follows directly from the definition of upper and lower semicontinuity for a family $\{J_\lambda\colon \lambda \in \Lambda\}$.

Lemma 4.2. *Let* X *be a Banach space,* Λ *be a topological space and* $J_\lambda \subset X$, $\lambda \in \Lambda$.

(i) *The family* $\{J_\lambda\colon \lambda \in \Lambda\}$ *is upper semicontinuous at* $\lambda = \lambda_0$ *if any sequence* $\{u_{\lambda_n}\}$, *with* $u_{\lambda_n} \in J_{\lambda_n}$, $n \in \mathbb{N}$, $\lambda_n \to \lambda_0$ *has a convergent subsequence to an element of* J_{λ_0}.
(ii) *If* J_{λ_0} *is compact and for any* $u \in J_{\lambda_0}$, *there exists a sequence* $\{u_{\lambda_n}\}$ *with* $u_{\lambda_n} \in J_{\lambda_n}$, $n \in \mathbb{N}$, $\lambda_n \to \lambda_0$ *such that* $u_{\lambda_n} \to u$, *then* $\{J_\lambda\colon \lambda \in \Lambda\}$ *is lower semicontinuous at* $\lambda = \lambda_0$.

Upper semicontinuity of attractors is a relatively simple matter to obtain now, being enough to use that the union of the attractors is relatively compact in X and that the family of semigroups behaves continuously with respect to ϵ. We will omit its proof since it is the same of the Dirichlet case proved in [8]:

Theorem 4.3. *The family of attractors* $\{\mathcal{A}_\epsilon\colon \epsilon \in (0, \epsilon_0]\}$ *is upper semicontinuous at* $\epsilon = 0$.

4.1. Continuity of the set of equilibria

The first task to show the lower semicontinuity of attractors is to obtain the continuity of the set of equilibria for (2.9). The fact that the operators A_ϵ are positive self-adjoint operators with compact resolvents plays an important role in the proof of continuity of the dynamics near equilibria through the fact that the linearization of the right-hand side of (2.9) has only a finite number of eigenvalues with positive real part.

Definition 4.1. The *equilibrium solutions* of (2.9), $\epsilon \in [0, \epsilon_0]$, are the solutions of the elliptic problems

$$A_\epsilon u - h(u) = 0. \tag{4.2}$$

We denote by $\mathcal{E}_\epsilon$ the set of solutions to (4.2), $\epsilon \in [0, \epsilon_0]$.

Definition 4.2. We say that an equilibrium u_* of (2.9) is *hyperbolic* if the spectrum $\sigma(A_\epsilon - h'(u_*))$ of $A_\epsilon - h'(u_*)$ is disjoint from the imaginary axis.

Hereafter we assume that all equilibrium points of (2.9) with $\epsilon = 0$ are hyperbolic.

Proposition 4.1. *If all equilibrium points of* (2.9) *with* $\epsilon = 0$ *are isolated then there is only a finite number of them. Any hyperbolic equilibrium point* u_* *of* (2.9) *with* $\epsilon = 0$ *is isolated.*

Proof. We first observe that the nonlinearity $h : X_0^{\frac{1}{2}} \to X_0^{-\frac{s}{2}}$, $\frac{1}{2} < s < 1$, is uniformly bounded:

$$\begin{aligned}\|h(u)\|_{X_0^{-\frac{s}{2}}} &\leqslant \sup_{\substack{\phi \in X_0^{\frac{s}{2}} \\ \|\phi\| \leqslant 1}} \int_\Omega f\big(u(x)\big)\phi(x)\,\mathrm{d}x + \sup_{\substack{\phi \in X_0^{\frac{s}{2}} \\ \|\phi\| \leqslant 1}} \int_\Gamma g\big(\gamma\big(u(x)\big)\big)\gamma\big(\phi(x)\big)\,\mathrm{d}x \\ &\leqslant C_f \sup_{\substack{\phi \in X_0^{\frac{s}{2}} \\ \|\phi\| \leqslant 1}} \|\phi\|_{L^2(\Omega)} + C_g \sup_{\substack{\phi \in X_0^{\frac{s}{2}} \\ \|\phi\| \leqslant 1}} \big\|\gamma(\phi)\big\|_{L^2(\Gamma)} \leqslant C_h,\end{aligned}$$

where C_1 and C_2 are embedding constants and $C_h = C_1 C_f + C_2 C_g$. Then the compact map

$$X_0^{\frac{1}{2}} \ni u \mapsto A_0^{-1} h(u) \in X_0^{1-\frac{s}{2}} \Subset X_0^{\frac{1}{2}}$$

takes $X_0^{\frac{1}{2}}$ in the ball $B_{X_0^{\frac{1}{2}}}(0, C_3 C_h)$, where C_3 is the embedding constant from $X_0^{1-\frac{s}{2}}$ in $X_0^{\frac{1}{2}}$. From Schauder's Theorem we obtain that (2.9) with $\epsilon = 0$ has at least one equilibrium solution.

Notice that $u \in \mathcal{E}_0$ is an equilibrium solution of (2.9) with $\epsilon = 0$ if and only if u is a fixed point of the map

$$\Phi(u) = -\big(A_0 - h'(u_*)\big)^{-1}\big(h'(u_*)u - h(u)\big)$$

and there is $\delta > 0$ such that $\Phi : \overline{B_{X_0^{\frac{1}{2}}}(u_*, \delta)} \to \overline{B_{X_0^{\frac{1}{2}}}(u_*, \delta)}$ is a contraction. In fact, if $u, v \in B := \overline{B_{X_0^{\frac{1}{2}}}(u_*, \delta)}$, then

$$\begin{aligned}\big\|\Phi(u) - \Phi(v)\big\|_{X_0^{\frac{1}{2}}} &= \big\|A_0^{-(\frac{1-s}{2})} A_0\big(A_0 - h'(u_*)\big)^{-1}\big[h(u) - h(v) - h'(u_*)(u - v)\big]\big\|_{X_0^{-\frac{s}{2}}} \\ &\leqslant C_4\big\|h(u) - h(v) - h'(u_*)(u - v)\big\|_{X_0^{-\frac{s}{2}}} \leqslant \frac{\delta}{2}\|u - v\|_{X_0^{\frac{1}{2}}},\end{aligned} \tag{4.3}$$

where we have used the continuous differentiability of h and chosen δ suitably small. Also $\Phi(u) - u^* = \Phi(u) - \Phi(u^*)$ shows that Φ takes B into itself. Thus, for δ suitably small, Φ has a unique fixed point in $\overline{B_{X_0^{\frac{1}{2}}}(u_*, \delta)}$. $\square$

With the convergence of the resolvent operators, we are able now to show the convergence of the resolvent operators $(A_\epsilon + h'(u^*))^{-1}$ to $(A_0 + h'(u^*))^{-1}$. This is an intermediate step that will allow us to show the continuity of the set of equilibria when all the equilibria of the limiting problem are hyperbolic.

To accomplish that we will need to study the family of operators $\{h'(u^\epsilon)\colon \epsilon \in (0, \epsilon_0]\}$ when u_ϵ $X_\epsilon^{\frac{1}{2}}$-converges to $u_0 \in X_0^{\frac{1}{2}}$.

Lemma 4.3. *For each* u^ϵ *and* u *in* $X_\epsilon^{\frac{1}{2}}$, *we have that*

$$\big\|h'(u^\epsilon) - h'(u)\big\|_{\mathcal{L}(X_0^{\frac{1}{2}}, X_\epsilon^{-\frac{s}{2}})} \leqslant C\big\|u^\epsilon - u\big\|_{X_\epsilon^{\frac{1}{2}}}$$

and, if $\{u^\epsilon\}$ *and* $\{z_\epsilon\}$ *are* $X_\epsilon^{\frac{1}{2}}$*-convergent sequences with limits u and z, respectively, in* $X_0^{\frac{1}{2}}$*, then*

$$h'(u^\epsilon)z^\epsilon = f'_\Omega(u^\epsilon)z^\epsilon + g'_\Omega(u^\epsilon)z^\epsilon \to f'_\Omega(u)z + g'_\Gamma(u)z = h'(u)z$$

in $X_\epsilon^{-\frac{s}{2}}$, $\frac{1}{2} < s < 1$, *as* $\epsilon \to 0$.

Proof. We have

$$\begin{aligned}
&\big\|h'(u^\epsilon)z - h'(u)z\big\|_{X_\epsilon^{-\frac{s}{2}}} \\
&\leqslant \sup_{\substack{\phi\in X_0^{\frac{1}{2}}\\ \|\phi\|\leqslant 1}} \int_\Omega \big|\big[f'(u^\epsilon(x)) - f'(u(x))\big]z(x)\phi(x)\big|\,dx + \sup_{\substack{\phi\in X_0^{\frac{1}{2}}\\ \|\phi\|\leqslant 1}} \int_\Gamma \big|\big[g'(\gamma(u^\epsilon(x))) - g'(\gamma(u(x)))\big]\gamma(z(x))\gamma(\phi(x))\big|\,dx \\
&\leqslant \sup_{\substack{\phi\in X_0^{\frac{1}{2}}\\ \|\phi\|\leqslant 1}} \bigg(\int_\Omega \big|f''\big(\theta(x)u^\epsilon(x) - (1-\theta(x))u(x)\big)\big|^2\big|u^\epsilon(x) - u(x)\big|^2\big|z(x)\big|^2\,dx\bigg)^{\frac{1}{2}} \|\phi\|_{L^2(\Omega)} \\
&\quad + \sup_{\substack{\phi\in X_0^{\frac{1}{2}}\\ \|\phi\|\leqslant 1}} \bigg(\int_\Gamma \big|g''\big(\gamma\big(\theta(x)u^\epsilon(x) - (1-\theta(x))u(x)\big)\big)\big|^2\big|\gamma(u^\epsilon(x) - u(x))\big|^2\big|\gamma(z(x))\big|^2\,dx\bigg)^{\frac{1}{2}} \big\|\gamma(\phi)\big\|_{L^2(\Gamma)} \\
&\leqslant C_1C_f \sup_{\substack{\phi\in X_0^{\frac{1}{2}}\\ \|\phi\|\leqslant 1}} \bigg(\int_\Omega \big|u^\epsilon(x) - u(x)\big|^2\big|z(x)\big|^2\,dx\bigg)^{\frac{1}{2}} \|\phi\|_{X_0^{\frac{1}{2}}} \\
&\quad + C_2C_g \sup_{\substack{\phi\in X_0^{\frac{1}{2}}\\ \|\phi\|\leqslant 1}} \bigg(\int_\Gamma \big|\gamma(u^\epsilon(x) - u(x))\big|^2\big|\gamma(z(x))\big|^2\,dx\bigg)^{\frac{1}{2}} \big\|\gamma(\phi)\big\|_{H^{s-\frac{1}{2}}(\Gamma)}.
\end{aligned}$$

Thus,

$$\begin{aligned}
\big\|h'(u^\epsilon)z - h'(u)z\big\|_{X_\epsilon^{-\frac{s}{2}}} &\leqslant C_3\big\|(u^\epsilon - u)^2\big\|_{L^2(\Omega)}\big\|z^2\big\|_{L^2(\Omega)} + C_4 \sup_{\substack{\phi\in X_0^{\frac{1}{2}}\\ \|\phi\|\leqslant 1}} \big\|\big(\gamma(u^\epsilon - u)\big)^2\big\|_{L^2(\Gamma)}\big\|\big(\gamma(z)\big)^2\big\|_{L^2(\Gamma)}\|\phi\|_{H^s(\Omega)} \\
&\leqslant C_5\big\|u^\epsilon - u\big\|_{X_\epsilon^{\frac{1}{2}}}\|z\|_{X_0^{\frac{1}{2}}} \xrightarrow{\epsilon\to 0} 0.
\end{aligned}$$

Similarly,

$$\begin{aligned}
\big\|h'(u^\epsilon)z_\epsilon - h'(u)z\big\|_{X_\epsilon^{-\frac{s}{2}}} &\leqslant \big\|h'(u^\epsilon)z_\epsilon - h'(u^\epsilon)z\big\|_{X_\epsilon^{-\frac{s}{2}}} + \big\|h'(u^\epsilon)z - h'(u)z\big\|_{X_\epsilon^{-\frac{s}{2}}} \\
&\leqslant \big\|h'(u^\epsilon)\big\|_{\mathcal{L}(X_\epsilon^{\frac{1}{2}}, X_\epsilon^{-\frac{s}{2}})}\|z_\epsilon - z\|_{X_\epsilon^{\frac{1}{2}}} + o(1) \xrightarrow{\epsilon\to 0} 0. \qquad \square
\end{aligned}$$

Since

$$A_\epsilon\big(A_\epsilon - h'(u^\epsilon)\big)^{-1} = I + h'(u^\epsilon)\big(A_\epsilon - h'(u^\epsilon)\big)^{-1},$$

the following holds:

Lemma 4.4. *Assume that* $\{u^\epsilon\}$ $X_\epsilon^{\frac{1}{2}}$*-converges to* $u \in X_0^{\frac{1}{2}}$ *and that* $0 \notin \sigma(A_0 - h'(u))$*. Then, for any* $0 \leqslant \theta \leqslant 1$*, there is* $\bar{\epsilon} \in (0, \epsilon_0]$ *such that*

$$\big\{(A_\epsilon)^\theta\big(A_\epsilon - h'(u^\epsilon)\big)^{-1}\colon\ 0 < \epsilon \leqslant \epsilon_0\big\} \subset \mathcal{L}(X)$$

is collectively compact, uniformly bounded and $(A_\epsilon)^\theta(A_\epsilon - h'(u^\epsilon))^{-1} \xrightarrow{\epsilon\to 0} (A_0)^\theta(A_0 - h'(u))^{-1}$ *(in the sense of Definition* 3.3*).*

Now we are ready to prove the convergence of the set of equilibria.

Proposition 4.2. *Suppose that u_* is an equilibrium solution for* (2.9) *with $\epsilon = 0$ and that $0 \notin \sigma(A_0 - h'(u_*))$. Then there are $\bar{\epsilon} > 0$ and $\delta > 0$ such that the problem* (2.9) *with $\epsilon \in (0, \bar{\epsilon})$ has exactly one equilibrium solution, u_*^ϵ, in $\{w^\epsilon \in X_\epsilon^{\frac{1}{2}}: \|w^\epsilon - u_*\|_{X_\epsilon^{\frac{1}{2}}} \leqslant \delta\}$. Furthermore, $\|u_*^\epsilon - u_*\|_{X_\epsilon^{\frac{1}{2}}} \to 0$ as $\epsilon \to 0$.*

For the proof we consider the operator

$$\Phi_\epsilon(u^\epsilon) = -\big(A_\epsilon - h'(u_*)\big)^{-1}\big(h'(u_*)u - h(u^\epsilon)\big)$$

in a small closed neighborhood of u^*. We observe that from Lemma 4.4 we have that, for $\frac{1}{2} < s < 1$,

$$\sup_{0 \leqslant \epsilon \leqslant \bar{\epsilon}} \big\|\big(A_\epsilon - h'(u_*)\big)^{-1}\big\|_{\mathcal{L}(X_\epsilon^{\frac{1}{2}}, X_\epsilon^{-\frac{s}{2}})} < \infty.$$

Now using Lemma 4.3, that is, the continuous differentiability of $h: X_\epsilon^{\frac{1}{2}} \to X_\epsilon^{-\frac{s}{2}}$, and the Uniform Contraction Principle implies the result.

Remark 4.1. Now that we have obtained a unique equilibrium point u_ϵ^* for (1.1) in a small neighborhood of the equilibrium point u^* for (1.2) we can consider the linearization $A_\epsilon - h'(u_\epsilon^*)$ and from the convergence of u_ϵ^* to u^* it is easy to obtain that $(A_\epsilon - h'(u_\epsilon^*))^{-1}$ converges compactly to $(A_0 - h'(u^*))^{-1}$. Consequently, the hyperbolicity of u^* implies (for suitably small ϵ) the hyperbolicity of u_ϵ^*. With this, we are ready to study the continuity properties of the flows near equilibria.

4.2. Continuity of local unstable manifolds

Let us study Eq. (2.9) in a neighborhood of the hyperbolic equilibrium u_*^ϵ, looking for a moment at a linear problem. If we consider the change of variables $v = u - u_*^\epsilon$, we have

$$\begin{cases} \dot{v} + \bar{A}_\epsilon v = h\big(v + u_*^\epsilon\big) - h\big(u_*^\epsilon\big) - h'\big(u_*^\epsilon\big)v, \\ v(0) = u_0^\epsilon - u^* = v_0^\epsilon \end{cases} \tag{4.4}$$

where $\bar{A}_\epsilon = A_\epsilon - h'(u_\epsilon^*)$. In this equation, for v very small, the nonlinear part is very small. It is natural then to consider what happens when we neglect the nonlinearity; that is, what happens to the equation

$$\begin{cases} \dot{v} + \bar{A}_\epsilon v = 0, \\ v(0) = v_0^\epsilon. \end{cases} \tag{4.5}$$

If $\bar{Q}_\epsilon^+$ is the projection defined by the spectrum of $\bar{A}_\epsilon$ to the right side of the imaginary axis, that is, by $\sigma_\epsilon^+ = \{\mu \in \sigma(-\bar{A}_\epsilon): \operatorname{Re}\mu > 0\}$, we have that for $v_0 \in \bar{Q}_\epsilon^+ X_\epsilon^{\frac{1}{2}}$ then the solution $v(t, v_0^\epsilon)$ of (4.5) exists for all negative time and $v(t, v_0^\epsilon) \to 0$ as $t \to -\infty$ and $v + u_*^\epsilon \to u_*^\epsilon$ as $t \to -\infty$. When we perturb (4.5) with a very small nonlinearity we should observe solutions of (4.4) that exist for all negative time. Of course the initial data for which such solutions exist will no longer be in $\bar{Q}_\epsilon^+ X_\epsilon^{\frac{1}{2}}$ but in a nonlinear manifold near it.

Notice that from Lemma 4.4 it follows that $\bar{A}_\epsilon^{-1}$ converges compactly to $\bar{A}_0^{-1}$ as $\epsilon \to 0$. Once it is established, we obtain similar versions of Lemma 3.2 and Theorem 3.3 to the operators $\bar{A}_\epsilon$.

Moreover, if we consider $\bar{A}_\epsilon^+$ and $\bar{A}_\epsilon^-$ the restrictions of $\bar{A}_\epsilon$ to $W_\epsilon^+ = \bar{Q}_\epsilon^+ X_\epsilon^{\frac{1}{2}}$ and $W_\epsilon^- = (I - \bar{Q}_\epsilon^+)X_\epsilon^{\frac{1}{2}}$, respectively, we also have analogous results for Lemma 3.2 and Theorem 3.3 to these operators, so we can show a version of Trotter–Kato Theorem for the family of linearized semigroups $\{e^{\bar{A}_\epsilon^+ t}: \epsilon \in [0, \epsilon_0]\}$. In fact, we have

Proposition 4.3. *Let $\{u_*^\epsilon: \epsilon \in (0, \epsilon_0]\}$ be a sequence of solutions of* (4.2) *such that $\{u_*^\epsilon\}$ $X_\epsilon^{\frac{1}{2}}$-converges to u_*. If $\bar{A}_\epsilon = A_\epsilon - h'(u_*^\epsilon)$, then there exists $\bar{\epsilon} > 0$ such that $\sigma(-\bar{A}_\epsilon)$ does not intercept the imaginary axis, for $\epsilon \in [0, \bar{\epsilon}]$, and $\|(A_\epsilon - h'(u_*^\epsilon))^{-1}\|_{\mathcal{L}(X_\epsilon^{-\frac{s}{2}}, X_\epsilon^{\frac{1}{2}})} \leqslant C$, with C independent of ϵ. Moreover, $\bar{Q}_\epsilon^+$ converges compactly to $\bar{Q}_0^+$ (in $\mathcal{L}(X_\epsilon^{-\frac{s}{2}}, X_\epsilon^\theta)$, $\theta \in [-\frac{s}{2}, \frac{1}{2}]$) as $\epsilon \to 0$ and the family of sets σ_ϵ^+ is upper and lower semicontinuous at $\epsilon = 0$. Furthermore, there are $\beta > 0$ and $M \geqslant 1$ such that, for $\epsilon \in [0, \epsilon_0]$, we have*

$$\begin{aligned} &\big\|e^{-\bar{A}_\epsilon t}\bar{Q}_\epsilon^+\big\|_{\mathcal{L}(X_\epsilon^{-\frac{s}{2}})} \leqslant M e^{\beta t}, \quad t \leqslant 0, \\ &\big\|e^{-\bar{A}_\epsilon t}\big(I - \bar{Q}_\epsilon^+\big)\big\|_{\mathcal{L}(X_\epsilon^{-\frac{s}{2}}, X_\epsilon^{\frac{1}{2}})} \leqslant M t^{-\frac{(1+s)}{2}} e^{-\beta t}, \quad t > 0. \end{aligned} \tag{4.6}$$

Next we study the continuity of local unstable manifolds. Assume that $\{u_*^\epsilon\}_{\epsilon \in [0, \epsilon_0]}$ is a sequence of solutions of (4.2) with $\|u_*^\epsilon - u_*\|_{X_\epsilon^{\frac{1}{2}}} \to 0$ as $\epsilon \to 0$.

The technic that we utilize to obtain the continuity of unstable manifolds is write them as a graphic $W^u_\epsilon = \{(z, S_\epsilon z)\colon z \in \bar{Q}^+_\epsilon X^{\frac{1}{2}}_\epsilon\}$ and to observe that the functions $S_\epsilon : \bar{Q}^+_\epsilon X^{\frac{1}{2}}_\epsilon \to (I - \bar{Q}^+_\epsilon) X^{\frac{1}{2}}_\epsilon$ behave continuously respect to parameter ϵ. Next we give an idea about how set up the functions S_ϵ. For more details we refer to [3] where the result is proved using the ideas of [15, §6.1].

First we decompose the space $X^{\frac{1}{2}}_\epsilon$ through the projection $\bar{Q}^+_\epsilon$, that is, $X^{\frac{1}{2}}_\epsilon = \bar{Q}^+_\epsilon X^{\frac{1}{2}}_\epsilon \oplus \bar{Q}^-_\epsilon X^{\frac{1}{2}}_\epsilon$, where $\bar{Q}^-_\epsilon = I - \bar{Q}^+_\epsilon$. This induces a decomposition of Eq. (2.9) in the following sense: if w is a solution of (2.9), we write $z^+ = \bar{Q}^+_\epsilon z$ and $z^- = \bar{Q}^-_\epsilon z$, and then

$$\begin{cases} \dot{z}^+ + \bar{A}^+_\epsilon z^+ = H_\epsilon(z^+, z^-), \\ \dot{z}^- + \bar{A}^-_\epsilon z^- = G_\epsilon(z^+, z^-), \end{cases} \tag{4.7}$$

where $H_\epsilon(z^+, z^-) = \bar{Q}^+_\epsilon(h(z^+ + z^- + u^\epsilon_*) - h(u^\epsilon_*) - h'(u^\epsilon_*)(z^+ + z^-))$, $G_\epsilon(z^+, z^-) = \bar{Q}^-_\epsilon(h(z^+ + z^- + u^\epsilon_*) - h(u^\epsilon_*) - h'(u^\epsilon_*)(z^+ + z^-))$ are continuously differentiable with $H_\epsilon(0,0) = G_\epsilon(0,0) = 0$ and $H'_\epsilon(0,0) = G'_\epsilon(0,0) = 0$. Since W^u_ϵ is invariant, for an initial data $(\tau, S_\epsilon(\tau)) \in W^u_\epsilon$, the solution of (4.7) stays in the graphic of S_ϵ for all $t \in \mathbb{R}$. This ensures that $z^-(t) = S_\epsilon(z^+(t))$ and, for all t, Eq. (4.7) can be rewritten as

$$\begin{cases} \dot{z}^+ + \bar{A}^+_\epsilon z^+ = H_\epsilon\big(z^+, S_\epsilon(z^+)\big), \\ \dot{z}^- + \bar{A}^-_\epsilon z^- = G_\epsilon\big(z^+, S_\epsilon(z^+)\big). \end{cases} \tag{4.8}$$

Furthermore, the solution $(z^+(t), z^-(t))$ must go to 0 as $t \to -\infty$ must go to zero as $t \to -\infty$ and in particular it must stay bounded. Since

$$z^-(t) = \mathrm{e}^{-\bar{A}^-_\epsilon (t-t_0)} z^-(t_0) + \int_{t_0}^{t} \mathrm{e}^{-\bar{A}^-_\epsilon (t-s)} G_\epsilon\big(z^+(s), S_\epsilon\big(z^+(s)\big)\big)\, \mathrm{d}s,$$

making $t_0 \to -\infty$, we have that

$$z^-(t) = S_\epsilon\big(z^+(t)\big) = \int_{-\infty}^{t} \mathrm{e}^{-\bar{A}^-_\epsilon (t-s)} G_\epsilon\big(z^+(s), S_\epsilon\big(z^+(s)\big)\big)\, \mathrm{d}s,$$

and in particular

$$S_\epsilon\big(z^+(\tau)\big) = z^-(\tau) = \int_{-\infty}^{\tau} \mathrm{e}^{-\bar{A}^-_\epsilon (\tau-s)} G_\epsilon\big(z^+(s), S_\epsilon\big(z^+(s)\big)\big)\, \mathrm{d}s.$$

Thus, we should have S_ϵ as a fixed point of an operator defined in a space of adequate functions. Once this is accomplished, the convergence of linear unstable give us the continuity if nonlinear unstable manifolds, with a similar argument as it is done in [3,8].

Proposition 4.4. *Assume that u_* is a hyperbolic equilibrium for* (1.2). *Proposition* 4.2 *guarantees that the problem* (1.1) *has a unique equilibrium u^ϵ_* in a small neighborhood of u_*. Then there exist $\delta > 0$ and $\bar{\epsilon} \in (0, \epsilon_0]$ such that u^ϵ_* has an unstable local manifold $\mathrm{W}^u_{\mathrm{loc}}(u^\epsilon_*) \subset X^{\frac{1}{2}}_\epsilon$, for $\epsilon \leqslant \bar{\epsilon}$, and if we denote*

$$\mathrm{W}^u_{\delta,\epsilon}\big(u^\epsilon_*\big) = \big\{ w \in \mathrm{W}^u\big(u^\epsilon_*\big),\ \big\| w - u^\epsilon_* \big\|_{X^{\frac{1}{2}}_\epsilon} < \delta \big\}, \quad \epsilon \in [0, \bar{\epsilon}],$$

then $\mathrm{W}^u_{\delta,\epsilon}(u^\epsilon_*)$ converges to $\mathrm{W}^u_{\delta,0}(u_*)$ as $\epsilon \to 0$, that is,

$$\sup_{w_\epsilon \in \mathrm{W}^u_{\delta,\epsilon}(u^*_\epsilon)} \inf_{w \in \mathrm{W}^u_{\delta,0}(u_*)} \|w_\epsilon - w\|_{X^{\frac{1}{2}}_\epsilon} + \sup_{w \in \mathrm{W}^u_{\delta,0}(u_*)} \inf_{w_\epsilon \in \mathrm{W}^u_{\delta,\epsilon}(u^\epsilon_*)} \|w_\epsilon - w\|_{X^{\frac{1}{2}}_\epsilon} \to 0, \quad \textit{as } \epsilon \to 0.$$

4.3. Lower semicontinuity of attractors

Since $\{T_\epsilon(t)\colon t \geqslant 0\}$ is a gradient semigroup, as we can see in [7], the attractor $\mathcal{A}_\epsilon$ can be characterized as the unstable manifold of the equilibrium set $\mathcal{E}_\epsilon$; that is, as

$$\mathcal{A}_\epsilon = W^u(\mathcal{E}_\epsilon).$$

Moreover, if all equilibria for (2.9) with $\epsilon = 0$ are hyperbolic, it follows, from Proposition 4.1, there are only finitely many of them and $\mathcal{A}_\epsilon = \bigcup_{u^\epsilon_* \in \mathcal{E}_\epsilon} \mathrm{W}^u(u^\epsilon_*)$, that is, the attractors $\mathcal{A}_\epsilon$ consist of the union of the unstable manifolds of equilibrium solutions.

Theorem 4.4. *(See Fig. 1.) The family of attractors $\{\mathcal{A}_\epsilon\colon \epsilon \in (0, \epsilon_0]\}$ is lower semicontinuous at $\epsilon = 0$.*

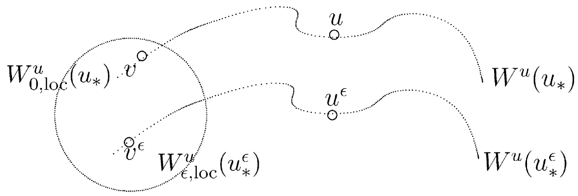

Fig. 1. Lower semicontinuity of attractors.

Proof. Let $u \in \mathcal{A}_0$. As $T_0(t)$ is a gradient system, we have $\mathcal{A}_0 = \bigcup_{w_* \in \mathcal{E}_0} \mathrm{W}^u(w_*)$, and then $u \in \mathrm{W}^u(u_*)$, for some $u_* \in \mathcal{E}_0$. Let $\tau \in \mathbb{R}$ and $v \in \mathrm{W}^u_{\delta,0}(u_*)$ such that $T_0(\tau)v = u$. Let u^ϵ_* such that $u^\epsilon_* \to u_*$. From convergence of unstable manifolds there is a sequence $\{v^\epsilon\}$, $v^\epsilon \in \mathrm{W}^u_{\delta,\epsilon}(u^\epsilon_*)$ such that $v^\epsilon \to v$ as $\epsilon \to 0$. Finally, from continuity of nonlinear semigroups, we obtain $T_\epsilon(\tau)v^\epsilon \to T_0(\tau)v = u$. To conclude we use Lemma 4.2 and observe that if $u^\epsilon = T_\epsilon(\tau)v^\epsilon$, then $u^\epsilon \in \mathcal{A}_\epsilon$, and u^ϵ $X_\epsilon^{\frac{1}{2}}$-converges to u. $\square$

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